Helsinki University of Technology Control Engineering Laboratory

Espoo 2003 Report 136

MODELLING, ANALYSIS AND CONTROLLER DESIGN OF TIME-VARIABLE FLOW PROCESSES

Kai Zenger

TEKNILLINEN KORKEAKOULU
TEKNISKA HÖGSKOLAN
HELSINKI UNIVERSITY OF TECHNOLOGY
TECHNISCHE UNIVERSITÄT HELSINKI
UNIVERSITE DE TECHNOLOGIE D´HELSINKI

Helsinki University of Technology Control Engineering Laboratory

Espoo March 2003 Report 136

MODELLING, ANALYSIS AND CONTROLLER DESIGN OF TIME-VARIABLE FLOW PROCESSES

Kai Zenger

Dissertation for the degree of Doctor of Science in Technology to be presented with due permission of the Department of Automation and Systems Technology, for public examination and debate in Auditorium T2 at Helsinki University of Technology (Espoo, Finland) on the 11th of April, 2003, at 12 noon.

Helsinki University of Technology

Department of Automation and Systems Technology

Control Engineering Laboratory

Distribution:
Helsinki University of Technology
Control Engineering Laboratory
P.O. Box 5400
FIN-02015 HUT, Finland
Tel. +358-9-451 5201
Fax. +358-9-451 5208
E-mail: control.engineering@hut.fi
http://www.control.hut.fi/

ISBN 951-22-6438-2 (print)
ISBN 951-22-6439-0 (pdf)
ISSN 0356-0872

Picaset Oy
Helsinki 2003

Available on net at http://lib.hut.fi/Diss/2003/isbn9512264390

HELSINKI UNIVERSITY OF TECHNOLOGY	ABSTRACT OF DOCTORAL DISSERTATION

Author: Kai Zenger
Name of the dissertation:
Modelling, Analysis and Controller Design of Time-Variable Flow Processes
Language of the dissertation: English
Translation in Finnish/English: Aikavarianttien virtausprosessien mallinnus, analyysi ja säätösuunnittelu.
Date of the manuscript: March 17, 2003
Monograph

Department: Automation and Systems Technology
Laboratory: Control Engineering Laboratory
Field of research: Control Engineering
Supervisor *Professor* Heikki Koivo., *AS-74*

Pre-examiners : Kurt-Erik Häggblom, Alf Isaksson
Opponent : Pentti Lautala, Alf Isaksson

A systematic theory for analysis and controller design of material transport systems under unsteady flow conditions is developed. It is assumed that the system is linear with respect to material concentrations so that the input-output dynamics can be characterized by a time-varying weighting function. The relation between the residence time distribution function and the weighting function is derived, and it is shown that the two functions become equal, when represented as functions of a new integrated time variable. A considerable complexity reduction is achieved, if, additionally, the weighting function becomes invariant with respect to the new time scale (volumetric scale).

It is shown that systems consisting of a series of perfect mixers with possible bypass flows and recycling is invariant with respect to the volumetric scale. A similar result applies to time variable delays, which become constants in the new time scale.

Structural properties i.e. stability, controllability and observability are shown to be unchanged in the transformation thus making it possible to use analysis and synthesis methods of classical control theory of linear time-invariant systems. By this way, a time-variable PID controller and LQ controller are derived and tested. As a special result it is shown that a PID-controller with time-variable coefficients can stabilize a system, which would be unstable in the case of varying flow rates, if a controller with constant coefficients were used.

The theoretical results and controller performance are tested by simulations and practical tests carried out by a laboratory-scale pilot plant. The results are shown to be in excellent agreement with those predicted in the theory.

Keywords
Time-varying systems, weighting function, residence time distribution, process control, continuous flow systems, variable flow, variable volume.

Distributed by / Publisher
Helsinki University of Technology, Control Engineering Laboratory
ISBN 951-22-6438-2 (print)
ISSN 0356-0872 Pages. 118

Contacts of the author
p. +358-9-4515204, email. kai.zenger@hut.fi

The dissertation can be read at http://lib.hut.fi/Diss/

Preface

The thesis was carried out in the Control Engineering Laboratory at Helsinki University of Technology. I wish to express my sincere gratitude to Professor Antti J. Niemi, who introduced me to the topic and was always helpful during long discussions concerning the difficult points in the theory. Professor Heikki Koivo was not involved in this research, but his character, humour and encouragement have had a great role especially in the last period of the work. Discussions with Professor Raimo Ylinen and Dr. Jussi Orava on the theoretical part have been invaluable on many occasions. Several persons helped me during the test phase–especially in the use of the pilot plant and in the preparation of tracer tests by radioactive tracers. The assistance of Dr. Jovan Thereska, Dr. Jose Griffith Martinez, Dr. Harri Kytömaa and Mr. Pekka Viitanen during different phases of the tests has been of great help. Mr. Erkki Solin and Mr. Jari-Pekka Ruonio did a good job in the construction and maintenance of the test equipment and the process vessels.

With warmth and gratitude I remember those long working hours and discussions with Docent Pentti Jutila, who was a good colleague and friend. He taught me all I know about pH control and much about other process control issues as well. Without his help the practical control tests with the pilot-plant would have been much more elaborate, if not impossible. Pentti passed away unexpectedly during Christmas time a year ago. His expertise and personality will be missed.

During the research financial support has been provided by the Academy of Finland, by the Jenny and Antti Wihuri Foundation, by Suomen Kulttuurirahasto and by Automaatiosäätiö. This support is gratefully acknowledged.

Finally, I thank the whole personnel of the Control Engineering Laboratory. *Most of you have come to the laboratory after me – some have already left for other jobs and duties – but these years have been a lot of fun!*

Espoo, March 17, 2003

Kai Zenger

Contents

Symbols and abbreviations

The main symbols and notations used in the text are summarized below in alphabetical order. The section or chapter, where each symbol has appeared for the first time, is also mentioned. 'Local' variables or functions have not been included in the list. The meaning of the variables and functions have also been explained in the main text, when used for the first time.

Symbols		**Section or Chapter**
A	Coefficient matrix in state-space representation	3.1
a_k, a	Volume ratio of mixers in series	3.4
B	Coefficient matrix in state-space representation	3.1
C	Coefficient matrix in state-space representation	3.1
c_c	Reagent concentration in a test system	6.2
c_i	Input concentration	2.2
c_k	Static gain of reference in state feedback control	6.3
c_o, c	Output concentration	2.2
$c_0, c_1, c_2, ...$	Intermediate concentrations	3.3
c_{pi}	Output concentration of an ideal mixer without dynamics	6.2
D	Derivative part in the PID controller output	6.1
D	Coefficient matrix in state-space representation	3.1
d_1, d_2	Constant coefficients in the function defining the modified time scale	3.1
dM_A	Amount of component A in flow	2.2
dM	Volumetric flow element	2.2
$dt, d\tau$	Infinitesimal time intervals	2.2
E	Residence time distribution function; same as p	2.4
E	System matrix in the transformed state representation	7.2
e	Control error (reference-system output)	6.1
F	Matrix in the transformed state representation	7.2

f	Function which defines the volumetric scale	2.4
f_t	Restriction function corresponding to the scale z_t	4.5
G	Matrix in the transformed state representation	7.2
g	Weighting function (same as p')	2.1
H	Matrix in the transformed state representation	7.2
h	Inverse function of f	2.4
I	Integration part in the PID controller output	6.1
i	Time index	3.3
J	Cost in LQ control	6.3
j	Time index	3.4
K_{pz}, K_{iz}, K_{dz}	Tuning coefficients of the PID controller	6.1
K	Proportional gain in PID controller	6.1
K	Observer gain	6.4
k	Scalar function causing time-variability in the system realization	3.1
k	Time index	3.3
k_t	Fixed value of function k at time t	6.4
L	Controller coefficient in state feedback control	6.4
l_k	Flow and volume ratio of mixers in series	3.4
M	Amount of material	2.3
M_A, M_E	Observability gramian	7.2
M_n	Class of Lyapunov transformation matrices	7.2
m	Number of inputs	3.1
N	'Strength' of the impulse	2.3
N	Number of points in the discretized impulse response	5.3
N	Coefficient in the lag part of the modified PID controller algorithm	6.1
n	Number of states	3.1
n	Number of vessels in series	3.3
P	Proportional part in the PID controller output	6.1
P	Transformation between the state variables	7.2
p'	Weighting function (same as g)	2.2
p	Residence time distribution	2.2
Q_0	Nominal process flow rate in a test system	6.2
Q_i	Input flow rate	2.2
Q_o	Output flow rate	2.2
$Q_1, Q_2, ...$	Intermediate flow rates	3.3

Q_{pi}	Input flow rate of a test system	6.2
Q_{po}	Output flow rate of a test system	6.2
Q_c	Reagent flow rate in a test system	6.2
R	Weight (matrix) related to the input variable in LQ control	6.3
r	Number of outputs	3.1
r	Absolute time instant, in which a particle leaves the vessel	4.2
r	Reference variable	6.3
R_1, R_2	Abbreviations for two corresponding state representations in time and z-domains	3.2
S	Variable (matrix) in the Riccati equation	6.3
$S(t_f)$	Weight (matrix) related to the cost of the final state in LQ control	6.3
S	Symbolic representation of a system	7.1
s	Complex variable used in the Laplace transformation	4.1
T	Sampling interval	6.1
T, T_0	Specified time intervals	4.1
T_z	Specified time interval in z-domain	4.1
T_d	Delay function	4.1
T_{dap}	Approximation of the delay function	4.2
T_i	Integration time in PID controller	6.1
T_d	Derivation time in PID controller	6.1
t	Time variable	2.1
t_f	Final time in LQ cost criterion	6.3
t_0	Initial time	2.1
t_1	A specified time point	4.1
$\bar{t}$	Mean residence time	2.4
U_B, U_F	Matrices in the SVD decomposition	7.4
u	Input function	2.1
u	Control variable	6.1
u_i	Component of the input variable	4.3
$\bar{u}, \bar{v}$	Augmented input variables	7.4
V	Liquid volume	2.2
$V_1, V_2, ...$	Intermediate liquid volumes	3.3
V_K	Total liquid volume	3.3
V_B, V_F	Matrices in the SVD decomposition	7.4
W_A, W_E	Controllability gramian	7.2
X	Weight (matrix) related to the state variable in LQ control	6.3

x	State variable	3.1
x_0	Initial state	3.1
x_e	Equilibrium point in state space	3.2
x_i	Component of the state variable	3.3
$\hat{x}$	Estimated value of state	6.4
$\tilde{x}$	Error between the real and estimated state value	6.4
y	Output function	2.1
z	Modified time variable, volumetric scale	1
z_c	Constant delay in volumetric scale	4.1
z_0	Initial time in the volumetric scale	2.4
z_1	A specified time point in the volumetric scale	4.2
z_t	A modified volumetric scale, which corresponds the restriction function	4.5
Δt	Discretization interval	5.3
Δz	Discretization interval in the volumetric scale	5.3
δ	The Dirac delta function	4.1
δ_τ	Unit impulse entering at time $t = \tau$	2.3
η	Absolute time instant, in which a particle enters a vessel	4.2
Ω	Set of admissible input functions	7.1
Φ	State transition matrix	3.2
$\varphi(t)$	Continuous test function	7.3
Σ, Σ_1	Symbolic representation of a system	7.1
Ξ_B, Ξ_F	Matrices in the SVD decomposition	7.4
θ	Scaled time variable (stationary conditions)	1
$\Vert \cdot \Vert$	Euclidean vector norm or the corresponding induced matrix norm	3.2
$\bar{(\cdot)}$	the 'bar' notation is used to indicate that the variable is expressed in the volumetric scale, e.g. $\bar{c}(z) = c(t)$	

Abbreviations

BIBO	Bounded input, bounded output	3.2
C-curve	Output response to an impulse function of tracer	1
E-curve	External Age Distribution (= RTD)	1
F-curve	Output response to a step function of tracer	1
FIR	Finite Impulse Response	1
I-curve	Internal Age Distribution	1
LQ-control	Linear Quadratic optimal control	1
pH	logarithmic measure of acidity	1
PID controller	Proportional Integral Derivative controller	1
PLC	Programmable Logic Controller	5.1
RTD	Residence Time Distribution	1
SISO	Single input - single output	2.1
SVD	Singular value decomposition	7.4

Chapter 1

Introduction

Control design of complex industrial plants is usually based on analysis of unit operations. The reason for that is the fact that the model structure of idealized small process entities is usually much simpler than the structure of larger process blocks. The total operation of the plant can then be analysed by combining the basic models together to represent larger entities of operation in the plant. If the connections between the unit models are defined properly, the analysis and control design of the whole plant becomes feasible.

One of the most elementary unit operations in process industry is mass transport, which has the purpose of moving material in a pipe from, say, one batch process to another. An important case is the continuous production line, in which material is pumped continuously through the process, and the unit operations take place in vessels like mixing tanks, chemical reactors etc. A traditional modelling technique is to describe the process as a combination of basic idealized models like perfect mixers and plug flow vessels, which may contain dead space and bypass or recycle flows (Levenspiel, 1962).

The dynamics of a continuous flow process is dependent on the mass flow rate. In a mixing tank the time constant of the process is determined by the flow rate through the vessel and the liquid volume in it. In traditional design the process is usually assumed to be in a nominal operation point so that the flow rates and volumes are constant, but this assumption is not always valid. Because of disturbances and intentional changes in the production rate the flow rate through a process entity is not always constant. The purpose of this text is to develop a systematic and mathematically sound theory, which can be used in the analysis and controller design of processes having such characteristics.

The origin of the theory of *residence time distributions* (RTDs) is usually credited to Danckwerts (1953), who used population-balance methods in the modelling of flow and mixing dynamics in vessels. The key idea was to consider the macroscopic balance of the process and to define suitable models for the residence times of particles in the system. The concepts *internal age distribution*, *external age distribution* (residence time distribution), *intensity function*, C, I, F and E curves formed the basic building blocks of the theory, which is now considered classical, see e.g. (Levenspiel, 1962), (Seinfeld and Lapidus, 1974), (Himmelblau and Bischoff, 1968).

The most attractive feature of the above models, especially in the case of the RTD, is the conceptually straightforward experimental determination of them by using suitable *tracer testing*. For example, if the flow process is linear with respect to particle concentrations flowing through the system, and if the system is in stationary operating conditions, the RTD and impulse response of the system are actually identical (the only difference is that the area under the RTD curve is scaled to unity). By injecting a suitable tracer impulse-wise into the system and measuring its output concentration continuously thereafter gives the impulse response, from which the RTD is easily calculated.

It is noteworthy that although the residence time distribution theory is a well-established and even age-old theory today, in many unit processes in process industry the experimental determination of the RTD is still the only applicable method to get a model of the system. The inspection of the curve gives the engineer an immediate view of the process dynamics and possible anomalies like mean residence times, by-passing, dead-space etc. One step further is then to fit a structural model to the measured RTD curve; traditionally a combination of perfect mixers, plug flow reactors, by-pass and recycle flows have been used. The structural model can then be used in control design. However, in many cases the tracer tests are used in process analysis and process design only (Thereska *et al.*, 1996).

Both chemical and radioactive tracers are used in the modelling of processes in several industry branches e.g. in petroleum industry, mineral processing and waste-water treatment plants (Thereska, 2001). Today there exist specific software packages for proper data handling related to the tracer test, measurements, model generation and validation, see e.g. (Žitný and Thýn, 1996). The background in the methods and the related software is to fit a structural model to the measured RTD (Bazin and Hodouin, 1988).

The classical residence time distribution covers only the case of stationary operating conditions, i.e. the flow rate through the system and the liquid volume in the system are constant. However, there is a strong practical demand to consider processes under unsteady operating conditions also, because of disturbances and intentional changes in the process operation. To consider such systems with time-varying dynamics brings the classical RTD theory beyond its scope, and extensions to the theory are needed.

References concerning RTDs under unsteady operating conditions of the process are quite rare in the literature. The classical work by Nauman (1969) presents a systematic approach to define time-varying age functions for stirred tank reactors. Dickens *et al.*,

(1989) studied the RTDs of unsteady flows in a baffled tube. Nir (1973) studied the mixing dynamics of lakes in unsteady operating conditions and actually defined a modified time scale similar to that discussed later in this work. Fernández-Sempere *et al.*, (1995) considered the general problem of variable flow and volume in perfect mixers, plug flow reactors and vessels with dispersion. The paper contains a reasonably concise treatment of the problem, although some of the results in it have been known earlier, see e.g. (Niemi, 1977a), (Niemi, 1988), (Niemi, 1990).

The most systematic work in the theory related to time-varying RTDs and their applications in control has been done by Niemi and his co-workers. The origin of this work dates back to the paper (Niemi, 1977a), in which the basic theory of time-variable residence time functions and weighting functions in processes with variable flow were presented. The same ideas – now including flow processes with variable liquid volume – were approached in (Niemi, 1981) and (Niemi, 1990). In these papers processes with different velocity profiles were covered, and fundamentals of time-variable controllers were presented. Some other papers can also be mentioned in this context, e.g. (Niemi, 1988), (Niemi, 1991).

The key idea of reducing the complexity of time-variable flow process models (assuming a constant velocity profile) is to use a modified time scale (z), and to represent the model equations with respect to this variable. The idea is a true extension to the use of the variable θ, which is known in classical RTD literature (Levenspiel, 1962), (Seinfeld and Lapidus, 1974). The θ scale is used when a flow system is in different stationary operating conditions; the flow rate is not allowed to vary continuously.

The theory developed further, when new researchers joined Niemi's group. The fundamental work of time-variable age distributions discussed earlier by Nauman (1969) was extended by Zenger (1995), who discussed and formulated the theory of time-variable RTDs, weighting functions and internal age distributions in a unified framework. The idea of using state-space representations to the analysis and controller design of time-variable flow processes was recognized at that time; for the main results of that approach, see (Zenger, 1992, 1993). Time-variable delays in plug flow vessels under variable flow rate mentioned already by Niemi (1977a) were modelled and analyzed by Zenger (1992), and more deeply by Zenger and Ylinen (1994). It is interesting to note that some of these results have connections to the work by Nihtilä (1991), who studied the adaptive control of processes with variable delays.

Applications of the theory have also been studied extensively by Niemi and his group. The idea of using a modified time scale to make the process model 'time-invariant' (with respect to the new scale) is useful in the sense that classical methods in the analysis and synthesis of dynamic systems become possible. Most of the work so far has been done in the fields of control and system identification.

The fact that the PID controller with constant coefficients in z-domain leads to a similar controller with variable coefficients in time domain was first noticed by Niemi, and this controller has been studied and used extensively, see e.g. (Niemi, 1991), (Niemi *et al.*, 1990), (Jutila and Jaakola, 1986), (Jutila *et al.*, 1999), (Zenger, 1992). The stability issues of the controller were studied by Zenger (1992, 1993). Practical tests were carried out and reported by Zenger *et al.* (1996). It is worth to mention that the algorithm in a slightly different form has been used by Jutila in several industrial projects. Also, in the report (Jutila, 1983) several adaptive algorithms in the control of pH processes were tested involving issues related to varying liquid flow.

Another idea in modelling has been to store the measured RTD as a time series only (in the scale z) and not to approximate it with a structural model. That idea has been elaborated on extensively by Tian (1994), see also (Tian *et al.*, 1992). In these studies an on-line identification method leading to a FIR model of the system (Wigren, 1990) was extended to the case of variable flow and variable volume. Robust control based on the identified model was considered by Tian (1994) and Niemi *et al.* (1997). An example of the loop-shaping control was presented by Niemi *et al.* (2001). Generally, it is somewhat surprising that the control applications related to models given by the RTD are few in the literature.

Apart from control viewpoint research results related to the general RTD theory can be found here and there in the literature. For example, Anderson and Pucar (1995) and Isaksson (1993) considered the estimation of the mean residence time of flow processes under unsteady flow; their method is an application of the theory of the modified time scale. Najim *et al.* (1996) considered the calculation of RTD of continuous flow nonlinear multivariable systems.

Additionally, there is a wide amount of literature concerning "standard" applications, in which the RTD has been measured to investigate the process behaviour. As an example of contemporary activities in tracer technology, see e.g. the Proceedings of the First International Congress Tracer and Tracing Methods, Nancy, France, 2001.

The outline of the thesis is as follows. In Chapter 2 the time varying residence time functions are defined. The starting point in the analysis is to consider material transport through a flow system and to model the input-output dynamics of the system. In this respect, the impulse response, weighting function and RTD are connected to each other in the case that the flow rate and the liquid volume are changing. The modified time scale (volumetric scale, z-scale) is introduced, and it is shown that under fairly general assumptions the weighting function and RTD coincide, when presented with respect to the new time scale.

Structural models of flow systems are discussed in Chapter 3. The input-state-output behaviour is considered by standard state-space representation, which can easily be formed e.g. by using models of a perfect mixer, mixers in series, mixers with bypass and recycle flows etc. The conditions under which the time-variability (changing flow rates and liquid volumes) of the representation can be changed invariant with respect to the modified time scale are proved. Structural properties, i.e. stability, controllability and observability are discussed, and they are shown to remain invariant in the transformation. A series of perfect mixers is used as an example, and the result, indicated already in Chapter 2, that the complexity reduction is possible in the case of changing flow rates but not generally under changing liquid volumes, is proved.

In Chapter 4 systems with variable time delays are connected to the previous theory. In system dynamics changing time delays arise naturally by considering plug flow vessels under changing flow rates and liquid volumes. It is shown that the variable delay changes into a constant, if the modified time scale is used. Again, that holds in the case of variable flow rate only; the case of variable liquid volume must be excluded. There are however ways to approach the variable volume case also and one alternative way is presented. The new concept of the *delay function* is introduced, and its properties are investigated.

Tracer experiments carried out with a laboratory-scale pilot plant are presented in Chapter 5 to verify the results obtained in previous chapters. Both chemical and radioactive tracers were used to three different kinds of vessels, which were subject to flow and liquid variations. The results showed an excellent match when compared to the results predicted by theory.

Controller design is the topic of Chapter 6. The idea is to start from a time-variable process model, change it into an invariant form with respect to the modified time scale, carry out controller design with classical techniques well-known for time-invariant systems, and finally transform the controller algorithm back into time domain. That procedure usually (but not always) leads to a controller with time variable coefficients. A good example is the time-variable PID controller, which is discussed first. By measuring the flow rate continuously the coefficients can be changed automatically such that e.g. the stability of the closed-loop system can be guaranteed. As a second example, LQ optimal control is discussed in connection to state feedback and state observer of time-variable systems.

In Chapter 7 the concept of a z-invariant system is formulated to put the results discussed earlier in a proper theoretical framework. Now the results are formulated in terms of general system theory without considering any process classes in particular. Linear time-variable state transformations are then considered as an alternative approach to changing the independent time variable. The motivation here is to develop and propose more general methods to analyse and control linear time-variable systems. Controller design by using LQ optimal control is used as an example.

Conclusions are given in Chapter 8.

The main contributions of the thesis can be summarized as follows:

- Material transport in flow systems with variable flow rates is considered, assuming that the velocity profile of particles going through the system does not change in spite of flow variations. If the system can be assumed linear with respect to input/output characteristics of material transport, the (time-varying) weighting function and residence time distribution function have a connection, which depends on the input and output flow rates, but not on the liquid volume in the system explicitly. The formula expressing the above relationship can be used to determine the residence time distribution from the measured impulse response data also in the case that the input and output flow rates have varied during the impulse test.

- When expressed as functions of the modified time scale (volumetric scale, z-scale) the weighting function and residence time distribution become equal in the case of variable flow rate. However, that does not hold if the liquid volume in the system also changes (input and output flow rates differ). The concept of the modified time scale has been known earlier, and it is not the invention of the author of the thesis.

- The theoretical framework is extended by considering state-space representations of flow systems, which arise naturally as models of perfect mixers, series of perfect mixers and possible bypass and recycle flows. The concept of a z-invariant realization is introduced to characterize a model, which can be changed into a constant-coefficient form by a change of the time-variable. The above system models are shown to be z-invariant in the case of variable flow rate. If the liquid volume is also changing, the system is z-invariant only in special cases.

- The structural properties of the system models (stability, controllability, observability) are shown to be invariant in the transformation. That makes the analysis and synthesis techniques of classical control theory of time-invariant systems feasible in the case of z-invariant systems. In other words, the design techniques are now on a mathematically sound basis. The theory is a true generalization to that of the scaled time variable, which is used in classical literature to model systems in different stationary operating conditions.

- The technique is then extended to cover time-variable delays caused by variable flow in plug flow vessels. The delay is described by the delay function, which has a complicated analytical form and which is difficult to calculate in closed form. However, with respect to the modified time scale the delay becomes constant, which is easy to deal with by classical methods. Systems of perfect mixers, plug flow vessels and bypass and recycle flows can now be modelled with state equations of constant coefficients and constant delay terms. Again, the delay becomes constant in the case of variable flow but not if the liquid volume is also changing. The latter case can be dealt with by using a new function (restriction function), but it is not analytically attractive. Its use would necessitate the storing of liquid volume values continuously; the system is not truly z-invariant, if values from different time scales (t and z) are mixed in this way.

- A time-variable PID controller and a time-variable linear quadratic optimal controller are presented as applications of the use of the theory. Stability results are discussed, and it is shown that a PID controller with constant coefficients does not necessarily stabilize a system under variable flow conditions, while a corresponding controller with variable parameters achieves a stable closed-loop system under all flow conditions. This phenomenon is also demonstrated by a practical pilot-plant test.

  The time-variable PID controller has been known earlier; it was not invented by the author. The LQ theory is also a classical one, but the application in z-invariant systems surprisingly leading in a constant feedback control law, is new.

- Examples describing the use of the technique are presented to verify the results. Tracer tests are used under variable flow conditions to determine the residence time distribution in time and z-domains, respectively. The theoretical result that the residence time distribution is invariant under the volumetric scale is verified.

- For comparison, a direct method of changing a time-varying system representation into a constant-coefficient form by a direct state transformation in time domain, is developed. However, this technique is still immature and it is presented for comparison purposes only. Theoretically, it has a wider application area than that discussed in the main part of the thesis.

- In short, the contribution of the work is to introduce the concept of z-invariant systems and to formalize it on a sound mathematical basis making both analysis and synthesis techniques tractable. Although the background behind the variable change has been known earlier, and the technique has been used in applications, the new formalism gives a much wider view on the subject, showing the possibilities and limitations clearly. For z-invariant systems controller design and analysis of the closed-loop operation are then straightforward by using the variety of tools available in the classical theory of time-invariant systems.

Chapter 2

Material Transport in Flow Processes

The basic theory for material transfer of mixing processes under unsteady operating conditions is developed in this chapter. Two different residence distributions are defined to characterize the residence time of particles in a flow system, in which the flow rate and liquid volume may be time-varying. The impulse response and weighting function are connected to the analysis to establish the input-output behaviour of the system with respect to concentrations. The volumetric scale is introduced in order to transform the time-varying system models into forms, which are invariant with respect to the new variable. Under the assumption of an invariant flow pattern the new scale means a complexity reduction, which can be seen as an extension to methods described in classical literature.

2.1 Input-output models of linear systems

Consider a single input–single output (SISO) system, which is assumed to be *linear*. The system can be described by the input-output model

$$y(t) = \int_{t_0}^{t} g(t,\tau)u(\tau)d\tau, \quad t \geq t_0 \tag{2.1}$$

where $y(\cdot)$, $u(\cdot)$ are the output and input functions, respectively, and $g(\cdot,\cdot)$ is the *weighting function*. The weighting function coincides with the *unit impulse response* of the system (from the time that the impulse enters the system). The system is assumed to be *causal*, which implies that $g(t,\tau) = 0$ for all $t < \tau$. Further, the system is *relaxed* at some time instant t_0, which means that the initial condition is zero (Chen, 1999). Sometimes the lower limit $-\infty$ in (2.1) is used in literature emphasizing that the system is relaxed initially. In this case the system must be assumed to be stable, because otherwise equation (2.1) will not give finite results. In what follows the symbol t_0 is used as a lower limit of the integral in (2.1) giving a natural time origin. For example, this time origin can be 0 or $-\infty$, whichever is convenient.

The system is generally *time-varying* implying that the value of the weighting function depends on two absolute time instants. In the case of a time-invariant system equation (2.1) can be written as the convolution integral

$$y(t) = \int_{t_0}^{t} g(t-\tau, 0)u(\tau)d\tau = \int_{t_0}^{t} g(t-\tau)u(\tau)d\tau \tag{2.2}$$

in which the usual convention $g(t-\tau, 0) \triangleq g(t-\tau)$ has been used for convenience.

In chemical modelling, tracer tests have been a practical method to determine the weighting pattern. The test is carried out by injecting an amount of a substance impulsewise into the vessel and by measuring the concentration continuously at the outlet of the vessel. When properly scaled the result gives the *residence time distribution* (RTD), which for linear and time-invariant systems coincides with the weighting function and unit impulse response of the system. Because the *transfer function* of the system is obtained as the Laplace transformation of the weighting function, the possibility for analysis and control design by classical methods is then apparent.

The situation is more complicated, if there are variations in the flow rate and liquid volume. The system and its model are then time-varying, which makes analysis much more involved, because the well-known classical theory of time-invariant systems in not usable anymore. The purpose of this chapter is to show, how the theory of weighting functions and RTDs can be extended to the time-varying case. A mathematical transformation is presented, which can be used to reduce the *complexity* of the resulting models into a form, where classical design methods again become possible.

2.2 Residence time functions

Consider a flow system with one input and one output. The liquid is assumed to be incompressible, and the system is closed meaning that all material entering the system with the flow will leave eventually. Furthermore, it is assumed that the process is linear with respect to concentrations.

The flow rate going through the system and the liquid volume in it may be time-varying, but the *flow pattern* is assumed to be invariant. That means that the statistical distribution of the flow elements going through the system remains the same despite of variations in the flow rate or liquid volume. Although the system model may contain time-varying coefficients the structure of the model describing the system does not change (Niemi, 1977a), (Zenger, 1995).

Consider a volumetric flow element $dM(\tau)$ entering the system in an infinitesimal time interval $[\tau, \tau + d\tau]$. (In short, the element enters at time τ.) Correspondingly, a volumetric flow element $dM(t)$ is assumed to leave the system during the time interval $[t, t + dt]$. The time that a single flow element stays in the system can be characterized by two distributions. The function $p(t, \tau)$ expresses the distribution of the residence times of particles *entering* at time τ, while $p'(t, \tau)$ characterizes the residence time of particles *leaving* the system at time t. Because of the possible variations in flow and volume, the absolute time instants (τ or t) are important here. This means that the system and its model are time-varying in contrast to the theory of time-invariant systems, for which only the difference of τ and t would be enough to characterize the input-output behaviour of the flow system (Levenspiel, 1962), (Seinfeld and Lapidus, 1974), (Himmelblau and Bischoff, 1968).

(Note that the apostrophe in $p'(t, \tau)$ above is used to make a distinction to the function $p(t, \tau)$; it does not mean differentiation. In this text the notation for the derivative is always d/dt or a 'dot' above a variable, e.g. $\dot{c}(t)$.)

Following the original derivation by Niemi (1977a), the residence functions and their relationship can be derived as follows. If the input and output flow rates are denoted by $Q_i(\cdot)$ and $Q_o(\cdot)$, respectively, the volumetric material equations for the entering and leaving volume elements at times τ and t are

$$dM(\tau) = Q_i(\tau)d\tau \tag{2.3}$$

$$dM(t) = Q_o(t)dt \tag{2.4}$$

The flow rates are assumed to be positive at each time instant. If the symbol $d[dM(t)]$ is used to express the amount in the leaving volume element which entered at the time τ, i.e. with the element $dM(\tau)$, the equation

$$d[dM(t)] = dM(\tau)p(t, \tau)dt = Q_i(\tau)d\tau\, p(t, \tau)dt \tag{2.5}$$

follows, in which the function $p(\cdot, \cdot)$ expresses the distribution of the residence times of particles entering at a fixed time instant τ. If the time t is considered, the entering time of particles τ is distributed. That can be expressed by the function $p'(\cdot, \cdot)$, and it follows that

$$d[dM(t)] = dM(t)p'(t, \tau)d\tau = Q_o(t)dt\, p'(t, \tau)d\tau \tag{2.6}$$

By combining (2.5) and (2.6)

$$Q_i(\tau)d\tau\, p(t, \tau)dt = Q_o(t)dt\, p'(t, \tau)d\tau \tag{2.7}$$

which simplifies to

$$p(t,\tau) = \frac{Q_o(t)}{Q_i(\tau)} p'(t,\tau) \tag{2.8}$$

Physical reasons give additional constraints to the two residence functions. Firstly, for all $t < \tau$ it holds $p(t,\tau) = p'(t,\tau) = 0$. Secondly,

$$\int_{\tau}^{\infty} p(t,\tau) dt = 1 \tag{2.9}$$

and

$$\int_{-\infty}^{t} p'(t,\tau) d\tau = 1 \tag{2.10}$$

which mean that all entering material will leave eventually and all leaving material has entered at some earlier time.

The above two constraint equations emphasize the probability density nature of the two residence functions. It is possible to interpret that the residence functions express the probability of the life time of a particle entering or leaving the system.

Analysis can also be carried out with respect to a fixed component of the process material. Consider a component A in the flow and denote its amount or mass M_A in the incoming or outgoing fluid. The momentary amount of material A in the incoming stream at time τ and in the outgoing stream at time t can be expressed as

$$dM_A(\tau) = Q_i(\tau) c_i(\tau) d\tau = c_i(\tau) dM(\tau) \tag{2.11}$$

$$dM_A(t) = Q_o(t) c(t) dt = c(t) dM(t) \tag{2.12}$$

in which $c_i(\cdot)$ and $c(\cdot)$ denote concentrations of the component A at the input and output, respectively.

If the amount of material A at the output at time t, which has entered at time τ, is denoted by $d[dM_A(t)]$, it follows that

$$d[dM_A(t)] = dM_A(\tau) p(t,\tau) dt = c_i(\tau) dM(\tau) p(t,\tau) dt = c_i(\tau) Q_i(\tau) d\tau \, p(t,\tau) dt \tag{2.13}$$

in which equations (2.3) and (2.11) have been used. The total amount of A in the effluent stream at time t can be calculated by summing all material components of A, which have entered earlier and are leaving at time t. Mathematically this means that

$$dM_A(t) = \int_{t_0}^{t} d[dM_A(t)] d\tau \tag{2.14}$$

By combining the equations (2.12), (2.13) and (2.14), it follows that

$$Q_o(t) c(t) dt = (\int_{t_0}^{t} c_i(\tau) Q_i(\tau) p(t,\tau) d\tau) dt \tag{2.15}$$

and finally

$$c(t) = \int_{t_0}^{t} c_i(\tau)\frac{Q_i(\tau)}{Q_o(t)}p(t,\tau)d\tau = \int_{t_0}^{t} p'(t,\tau)c_i(\tau)d\tau \tag{2.16}$$

From a system theoretic viewpoint the result shows that the residence function $p'(\cdot,\cdot)$ can be identified with the *weighting function* $g(\cdot,\cdot)$ of the system. The integral

$$c(t) = \int_{t_0}^{t} g(t,\tau)c_i(\tau)d\tau \tag{2.17}$$

where in this case

$$g(\cdot,\cdot) = p'(\cdot,\cdot) \tag{2.18}$$

is the well-known input-output relationship of a linear but possibly time-varying system, which is initially relaxed meaning that the initial conditions are zero.

Equation (2.8), which has also been used in (2.16), gives an interesting relationship between the two residence functions. The function $p(\cdot,\cdot)$ is called the *residence time distribution* (RTD) in classical literature (Seinfeld and Lapidus, 1974), (Levenspiel, 1962). The weighting function $p'(\cdot,\cdot)$ on the other hand is a basic concept in system theory as explained in any basic textbook of linear systems, see e.g. (Padulo and Arbib, 1974), (Chen, 1999). The importance of equation (2.8) lies in the fact that it becomes possible to conveniently use general system theory in the analysis of flow systems modelled by residence time distributions.

Additionally, there are two important points to notice in equation (2.8). Firstly, the possibly varying liquid volume does not appear explicitly in the equation. Instead, starting from mass balances and assuming a constant density, the equation

$$\dot{V}(t) = Q_i(t) - Q_o(t) \tag{2.19}$$

is obtained (Denn, 1986), which shows how the liquid volume depends on the incoming and outgoing flow rates. (It is assumed that the flow rates are such that the liquid volume is positive all the time.) Secondly, if the flow rate through the system is constant all the time, the two residence functions become the same, which confirms the well-known fact (Niemi, 1977a, 1988) that for time-invariant flow systems the residence time distribution and weighting function are equal. From the general system theory it is further known (Padulo and Arbib, 1974) that in the time-invariant case

$$g(t,\tau) = g(t-\tau,0) \stackrel{\triangle}{=} g(t-\tau) \tag{2.20}$$

which means that only the time difference between two time instants is important when characterizing the input-output behaviour of a linear time-invariant system. The same holds for the residence time distribution also

$$p(t,\tau) = p(t-\tau,0) \stackrel{\triangle}{=} p(t-\tau) \tag{2.21}$$

Actually, the term residence time distribution is mathematically incorrect, and e.g. residence time density function would be more appropriate. However, for historical reasons the phrase 'residence time distribution' (RTD) is here used, as in almost all literature discussing this concept.

2.3 Impulse response, RTD and the weighting function

There is an alternative approach to derive relationship (2.8) by using the well-known fact that in a linear system the weighting function and unit impulse response are equal from the time instant that the impulse has entered the system (Rugh, 1993). Let an amount M of material enter a system at time τ; for example, M moles of tracer is injected in a vessel impulsewise in a very short time interval. All material is assumed to leave eventually so that

$$M = \int_{t_0}^{\infty} Q_o(t)c(t)dt = \int_{t_0}^{\infty} Q_i(t)c_i(t)dt = \int_{t_0}^{\infty} Q_i(t)N\delta_\tau(t)dt = NQ_i(\tau) \tag{2.22}$$

where $c_i(\cdot)$ is the input concentration of the injected material and N expresses the 'strength' of the impulse. The units in the above equation are $[M] = mol$, $[Q_i] = [Q_o] = m^3/s$, $[c_i] = [c] = mol/m^3$, $[N] = mol \cdot s/m^3$. Equation (2.13) can be solved leading to

$$p(t,\tau)dt = \frac{d[dM_A(t)]}{dM_A(\tau)} \tag{2.23}$$

But the material A was injected impulsewise to the system at time τ; hence it holds that $d[dM_A(t)] = dM_A(t)$, and by equations (2.12) and (2.22) the result

$$p(t,\tau)dt = \frac{Q_o(t)c(t)dt}{M} = \frac{Q_o(t)c(t)dt}{\int_\tau^\infty Q_o(t)c(t)dt} \tag{2.24}$$

follows. Note that for constant flow rates the result simplifies to

$$p(t,\tau)dt = \frac{c(t)dt}{\int_\tau^\infty c(t)dt} \tag{2.25}$$

which is the classical expression for the RTD (Levenspiel, 1962), (Seinfeld and Lapidus, 1974).

Interpreting the impulse response (impulse entering at time τ) as the weighting function gives the output concentration (Zenger, 1995)

$$c(t) = Np'(t,\tau) \tag{2.26}$$

and further, by using (2.22), (2.24) and (2.26)

$$p(t,\tau) = \frac{Q_o(t)c(t)}{M} = \frac{Q_o(t)}{Q_i(\tau)}p'(t,\tau) \tag{2.27}$$

$(t \geq \tau)$. The result is the same as (2.8).

2.4 A modified time scale

The close relationship between the two residence functions $p(\cdot,\cdot)$ and $p'(\cdot,\cdot)$ leads to the natural question, whether there exists a transformation that would make the two functions equal. It would also be beneficial if the resulting input-output relationship could be expressed with a weighting function that would somehow have a 'time-invariant' nature. To study the existence of such a transformation let $p'(t,\tau)$ be the weighting function of the system. Let $z = f(t)$ where $f : t \mapsto f(t)$ is a positive, continuously differentiable and monotonously increasing function with the inverse $h : z \mapsto h(z)$ such that for $t \geq \tau$, $t = h(z)$ and $\tau = h(\xi)$. A straightforward change of variables in equation (2.16) leads to

$$c(h(z)) = \int_{z_0}^{z} p'(h(z),h(\xi))c_i(h(\xi))\dot{h}(\xi)d\xi \tag{2.28}$$

$(z_0 = f(t_0))$ which can be expressed as

$$\bar{c}(z) = \int_{z_0}^{z} \bar{p'}(z,\xi)\bar{c}_i(\xi)d\xi \tag{2.29}$$

in which for all t, τ and the corresponding z, ξ it holds that $\bar{c}(z) = c(h(z)) = c(t)$, $\bar{c}_i(\xi) = c_i(h(\xi)) = c_i(\tau)$, $\bar{p'}(z,\xi) = p'(h(z),h(\xi))\dot{h}(\xi)$. By using the well-known formula for the derivative of the inverse function $\dot{h}(\xi) = 1/\dot{f}(\tau)$ the relationship

$$\bar{p'}(z,\xi) = p'(t,\tau)/\dot{f}(\tau) \tag{2.30}$$

follows. If, additionally, for all t, τ

$$\frac{p'(t,\tau)}{\dot{f}(\tau)} = \bar{p'}(z-\xi,0) \tag{2.31}$$

the representation is invariant with respect to z, i.e.

$$\bar{p'}(z,\xi) = \bar{p'}(z-\xi,0) \tag{2.32}$$

The system is then called *z-invariant* with respect to the function $f(\cdot)$. The variable $z = f(t)$ can be interpreted as a new time variable, which transforms the system representation into a 'time-invariant' form.

Remark: Note that the system has been assumed to be relaxed initially, meaning that the initial values are assumed to be zero. As will be discussed in Chapter 7, the above definition should actually be *zero-state* z-invariant.

Note that corresponding to the normalization conditions (2.9) and (2.10) it follows that

$$\int_{\xi}^{\infty} \bar{p}(z,\xi)dz = 1 \tag{2.33}$$

$$\int_{-\infty}^{z} \bar{p'}(z,\xi)d\xi = 1 \tag{2.34}$$

in which

$$\bar{p}(z,\xi) = p(h(z),h(\xi))\dot{h}(z) = p(t,\tau)/\dot{f}(t) \tag{2.35}$$

and $\bar{p'}(z,\xi)$ is given by (2.30).

By using (2.30) and (2.35) the equation (2.8) can be written as

$$\bar{p}(z,\xi) = \frac{Q_o(t)}{Q_i(\tau)}\frac{\dot{f}(\tau)}{\dot{f}(t)}\bar{p'}(z,\xi) \tag{2.36}$$

and using the transformation

$$z = f(t) = \int_{t_0}^{t} \frac{Q_i(\nu)}{V(\nu)}d\nu \tag{2.37}$$

in which $V(\cdot)$ denotes the volume of the liquid in the system, it changes into

$$\bar{p}(z,\xi) = \frac{Q_o(t)}{Q_i(t)}\frac{V(t)}{V(\tau)}\bar{p'}(z,\xi) \tag{2.38}$$

The result shows that in the case of varying flow rates but constant volume

$$\bar{p}(z,\xi) = \bar{p'}(z,\xi) \tag{2.39}$$

meaning that the residence time distribution and weighting function are equal when represented as functions of the modified time scale. If the system is z-invariant, the above equation can be written as

$$\bar{p}(z,\xi) = \bar{p}(z-\xi,0) = \bar{p'}(z-\xi,0) \tag{2.40}$$

There are a few important observations to be made at this point. The concept of a z-invariant system is related to the chosen scale $z = f(t)$, so when the concept is used it is assumed that the scaling (or scaling function) has been given.

The fact that the RTD, weighting function and (unit) impulse response become equal in the volumetric scale makes an interesting analogy to the classical theory of linear systems and RTDs. However, the system representation is still in 'time-varying' form; if, additionally, equation (2.31) holds, the system becomes z-invariant making it possible to use analysis and synthesis methods of the classical and well-established theory of linear time-invariant control systems.

The varying flow rate is hidden under the transformation, which actually means that the complexity reduction is achieved by using a time-varying transformation. Note that the result (2.39) holds in the case of varying flow rates but not if the liquid volume is also changing i.e. if the input and output flow rates differ from each other.

Because the variable z is interpreted as a modified time, it must be monotonously increasing. In equation (2.37) it means that both the flow rate and liquid volume must be positive.

For a practical application consider the possibility to determine the RTD by using a tracer experiment (Zenger, 1995). When an amount of tracer is injected in a flow system impulsewise, the impulse response is obtained by measuring the output concentration as a function of time. That output concentration can be expressed by $c(t) = k_1 p'(t, \tau)$ in which k_1 is a constant. By considering equation (2.9) the RTD is then calculated by normalizing the area under the concentration curve to unity.

From equation (2.8) it is seen that the value of k_1 is indeed a constant, if the flow rate through the system (and thus the liquid volume also) is constant. However, if the flow rate is changing the measured impulse response must be multiplied by the output flow rate *before* normalization in order to get a correct RTD. So, the normalization is applied to the function $p_1(t, \tau) = Q_o(t)c(t)$ where $c(\cdot)$ is the measured response. In the stationary case the calculation of $p_1(t, \tau)$ first is not necessary, because scaling is then properly performed by the normalization procedure.

To demonstrate the above ideas and results an example case is considered.

Liquid is flowing through a vessel with a constant flow rate Q. The input and output flow rates are equal so that the liquid volume in the vessel is also a constant V. The volumetric scale is then

$$z = \int_0^t \frac{Q}{V} d\nu = \frac{Q}{V} t = \frac{t}{\bar{t}} \tag{2.41}$$

in which the time origin has been chosen to be zero for convenience. The term $\bar{t} = V/Q$ is the *mean residence time*. In the dimensionless volumetric scale the formulas for the weighting function and the RTD are

$$\bar{p'}(z, \xi) = \frac{p'(t, \tau)}{\dot{f}(\tau)} = \bar{t} p'(\bar{t} z, \bar{t} \xi) \tag{2.42}$$

$$\bar{p}(z,\xi) = \frac{p(t,\tau)}{\dot{f}(t)} = \bar{t}p(\bar{t}z,\bar{t}\xi) \tag{2.43}$$

If the system is time-invariant, it is z-invariant also so that

$$\bar{p'}(z,\xi) = \bar{t}p'(\bar{t}(z-\xi),0) \tag{2.44}$$

$$\bar{p}(z,\xi) = \bar{t}p(\bar{t}(z-\xi),0) \tag{2.45}$$

Moreover, the above two equations are equal, see (2.39). If the impulse is assumed to enter at time 0 ($\tau = 0$, $\xi = 0$), then

$$\bar{p'}(z,0) = \bar{p}(z,0) = \bar{t}p(\bar{t}z,0) \tag{2.46}$$

As an example consider a flow system under a steady operating condition $\bar{t}_1 = V_1/Q_1$, which has the RTD $p_1(t,0)$. Let the operating point be changed into another stationary point $\bar{t}_2 = V_2/Q_2$. If the flow system has an invariant flow pattern despite of the change in the operating point, the RTD $p_2(t,0)$ in terms of $p_1(t,0)$ can be calculated as follows. Choose a volumetric time scale $z = t/\bar{t}_2$. By using equation (2.46) the RTDs become

$$\bar{p}_1(z,0) = \bar{t}_1 p_1(\bar{t}_1 z,0)$$

$$\bar{p}_2(z,0) = \bar{t}_2 p_2(\bar{t}_2 z,0)$$

Because the system is z-invariant, the two functions must be equal, which means

$$p_2(t,0) = \frac{\bar{t}_1}{\bar{t}_2} p_1(\frac{\bar{t}_1}{\bar{t}_2}t,0) \tag{2.47}$$

or

$$p_1(t,0) = \frac{\bar{t}_2}{\bar{t}_1} p_2(\frac{\bar{t}_2}{\bar{t}_1}t,0) \tag{2.48}$$

In classical literature the variable θ

$$\theta = \frac{t}{\bar{t}} \tag{2.49}$$

has been used instead of z to define the *scaled time variable* with respect to which the RTD can be made constant under different steady operating conditions. Formula (2.46) is well-known to apply in *different* constant steady states, see e.g. (Himmelblau and Bischoff, 1968), (Levenspiel, 1962). Note that in the literature the variable E has usually been used to denote the RTD; equation (2.46) is then presented in the form

$$\bar{E}(\theta) = \tau E(t)\mid_{t=\tau\theta}$$

or even

$$E(\theta) = \tau E(t)$$

The above analysis shows that under the assumption of an invariant flow pattern the idea of using a scaled time variable can be extended to the case of continuously varying flow rates and liquid volumes. The use of the general volumetric scale z is then a clear extension to the existing RTD literature.

Chapter 3

Differential Systems

To proceed further in the development of the ideas presented in the previous chapter, some kind of a structure is needed to characterize the system model. A *differential system* is a natural candidate for several reasons. Firstly, flow processes are continuous systems, and it is therefore natural to use continuous time models. Secondly, in the classical literature and also in the practical modelling of chemical reactors today, such systems are modelled by combinations of perfect mixers, plug flow reactors and recycle flows. The models of these kinds of systems are easily available as a combination of differential equations. Thirdly, the effect of varying flow rates and liquid volumes in the model arises naturally in that the system structure does not change but the coefficients become time-variable instead. The basic assumption that the flow pattern does not change in spite of variations in flow and volume is then automatically contained in the process model. From the analysis point of view it is a big advantage that although the model becomes time-varying, it still remains linear.

It is not the purpose of this text to start from the theoretical definition of a *differential system* or *system* in general. For an extensive discussion on this topic (Zadeh and Desoer, 1963) provides a standard reference. Another starting point would be the *input-output representation* described e.g. by Blomberg and Ylinen (1983), or in a more general setting by Orava (1973, 1974).

In the current text a differential system means a system whose input and output are related by one or more ordinary differential equations (Zadeh and Desoer, 1963). In order for the model to be meaningful in technical perspective, it is assumed that the system has a unique solution through a given point, which is usually given as the initial condition. The differential equations are further transformed to a set of first order differential equations (state-space representation), and an output mapping provides the connection between the input, state and output variables. In what follows, when the term *representation* or *realization* is used, a state-space representation is generally meant.

3.1 State-space representations

Consider the state-space representation of a system

$$\begin{aligned} \dot{x}(t) &= A(t)x(t) + B(t)u(t) \qquad x(t_0) = x_0 \\ y(t) &= C(t)x(t) + D(t)u(t) \end{aligned} \tag{3.1}$$

in which the state $x(\cdot) \in (\Re^n)^{\Re}$, input $u(\cdot) \in (\Re^m)^{\Re}$, and output $y(\cdot) \in (\Re^r)^{\Re}$; the coefficients $A(\cdot) \in (\Re^{n\times n})^{\Re}$, $B(\cdot) \in (\Re^{n\times m})^{\Re}$, $C(\cdot) \in (\Re^{r\times n})^{\Re}$, and $D(\cdot) \in (\Re^{r\times m})^{\Re}$ are continuous functions on the real numbers $\Re$.

The *representation* is defined to be *z-invariant*, if a new time scale z exists such that the equations change into ones with constant coefficient matrices.

As in Chapter 2, let $z = f(t)$ where $f : t \mapsto f(t)$ is a continuously differentiable and monotonously increasing function with the inverse $h : z \mapsto h(z)$.

Proposition 1 *The state-space representation is z-invariant, if and only if it satisfies the following two conditions.*

1. *$A(t) = k(t)\bar{A}$ and $B(t) = k(t)\bar{B}$, where $k(t)$ is a positive and continuous scalar function, and $\bar{A}$ and $\bar{B}$ are constant matrices.*

2. *$C(t) = \bar{C}$ and $D(t) = \bar{D}$ are constant matrices.*

The transformation is then

$$z = f(t) = d_1 \int_{t_0}^{t} k(\nu)d\nu + d_2 \tag{3.2}$$

$d_1 > 0$, $d_2 \geq 0$ and the equivalent system

$$\begin{aligned} \frac{d\bar{x}(z)}{dz} &= (1/d_1)\bar{A}\bar{x}(z) + (1/d_1)\bar{B}\bar{u}(z) \qquad \bar{x}(z_0) = \bar{x}(f(t_0)) = x_0 \\ \bar{y}(z) &= \bar{C}\bar{x}(z) + \bar{D}\bar{u}(z) \end{aligned} \tag{3.3}$$

Proof: Consider the composite mapping $x(f(t))$. Because both $x(\cdot)$ and $f(\cdot)$ are differentiable functions, the chain rule can be applied in equation (3.1), which gives then

$$\begin{aligned} \frac{dx}{dz}(z)\frac{df}{dt}(t) &= A(t)x(h(z)) + B(t)u(h(z)) \\ y(h(z)) &= C(t)x(h(z)) + D(t)u(h(z)) \end{aligned} \tag{3.4}$$

Because z is monotonously increasing, $\dot{f}(t) > 0$ and

$$\begin{aligned} \frac{dx}{dz}(z) &= \frac{A(t)}{\dot{f}(t)}x(h(z)) + \frac{B(t)}{\dot{f}(t)}u(h(z)) \\ y(h(z)) &= C(t)x(h(z)) + D(t)u(h(z)) \end{aligned} \tag{3.5}$$

Because $\dot{f}(t)$ is a scalar function, it must hold that for all t $A(t)/\dot{f}(t)$, $B(t)/\dot{f}(t)$, $C(t)$ and $D(t)$ are constant matrices. Each element in matrices $A(\cdot)$ and $B(\cdot)$ must then have a common factor $k(\cdot)$. Equation (3.2) gives a general solution for $z = f(t)$. □

Remark 1: In Chapter 2 the concept of a z-invariant *system* was defined, in which the input-output relationship was invariant with respect to a scale z. Now the concept of a z-invariant *representation* is introduced. It is possible that a z-invariant system has two different state-space representations, one of which is z-invariant, while the other one is not. So it makes a difference, whether z-invariant *systems* or *representations* are discussed.

Remark 2: The fact that the coefficient matrices in (3.1) are assumed to be continuous functions guarantees that the system has a unique and continuously differentiable solution. Sometimes, e.g. in the case of switching systems, it is necessary to consider cases, in which the system matrix is only piecewise continuous. As discussed by Coddington and Levinson (1955) and Zadeh and Desoer (1963), let $\dot{x}(t) = A(t)x(t)$, in which $A(t)$ is a *regulated function* meaning that the matrix elements are piecewise continuous and have a right-hand limit at every point. Moreover, in each finite interval the amount of the discontinuity points is countable. Under these assumptions there is a unique and continuous function $x(t)$

$$x(t) = x_0 + \int_{t_0}^{t} A(\tau)x(\tau)d\tau$$

which satisfies the above differential equation except at the discontinuity points. For brevity, it is said that the continuous function $x(t)$ is the solution of the homogenous differential equation.

In the current text the possibility that the state equation may not have a solution at each time instant is avoided by assuming that the system matrix is continuous. The transformation $f(\cdot)$ is assumed to be even continuously differentiable; in practice that is implied by the transformation (3.2), $k(t)$ being continuous.

3.2 Structural properties

An important result given by the next two propositions states that the structural properties i.e. stability, controllability, and observability are preserved under the transformation (Zenger, 1993). For brevity, call the time-varying representation (3.1) R_1 and the corresponding representation with constant coefficients (3.3) R_2. Stability is considered with respect to the origin ($x_e = 0$). The equivalence between the two system representations is so strong that the following two results follow immediately. In fact, proving them feels like proving the inevitable.

Proposition 2 *If the function $z = f(t)$ that transforms R_1 into R_2 tends to infinity as $t \to \infty$, the following holds:*

1. *R_1 is (asymptotically) stable, if and only if R_2 is (asymptotically) stable.*
2. *R_1 is uniformly stable, if and only if R_2 is stable.*
3. *R_1 is uniformly exponentially stable, if and only if R_2 is asymptotically stable.*
4. *R_1 is (zero state) BIBO stable, if and only if R_2 is (zero state) BIBO stable.*

Proof: R_1 and R_2 are equivalent in the sense that $\bar{x}(z) = x(t)$, $\bar{u}(z) = u(t)$, $\bar{y}(z) = y(t)$, where the 'time' variables t and z have a one-to-one relationship $z = f(t)$, $t = h(z)$. If R_1 is stable, for every $\epsilon > 0$ there is a $\delta > 0$ such that $\|x(t_0)\| < \delta$ implies $\|x(t)\| < \epsilon$ for all $t \geq t_0$. The equivalence of the variables then guarantees that $\|\bar{x}(z_0)\| < \delta$ implies $\|\bar{x}(z)\| < \epsilon$, so that R_2 is stable as well. In a similar manner it can be proved that the stability of R_2 implies the stability of R_1.

For every interval $[t_0, t_1)$ there is a corresponding unique interval $[z_0, z_1)$ and vice versa. Moreover, $z_1 \to \infty$ as $t_1 \to \infty$. Hence, if the state variable of R_1 or R_2 approaches the origin as the 'time' variable tends to infinity, so does the state variable of the other representation. The asymptotic stability of one representation thus guarantees the asymptotic stability of the other as well.

For the second item note that the solutions of the state equations of R_1 and R_2 are the same, which in terms of the state transition matrices can be written as

$$\begin{aligned} \Phi_{R_1}(t,\tau) &= e^{\int_\tau^t A(\nu)d\nu} = e^{\bar{A}\int_\tau^t k(\nu)d\nu} \\ &= e^{(1/d_1)\bar{A}(z-\xi)} = \Phi_{R_2}(z,\xi) \end{aligned} \tag{3.6}$$

where $z = f(t), \xi = f(\tau)$. Now, a linear time-varying system is uniformly stable, if there exists a finite positive constant γ such that for any t_0 and x_0 the solution of the autonomous system satisfies

$$\parallel x(t) \parallel \leq \gamma \parallel x_0 \parallel, \quad t \geq t_0 \tag{3.7}$$

(Brockett, 1970), (Rugh,1993). Moreover, the state equation is uniformly stable if and only if for the state transition matrix it holds

$$\parallel \Phi(t,\tau) \parallel \leq \gamma \tag{3.8}$$

for all t, τ such that $t \geq \tau$, (Rugh, 1993). The proof of the second item in the proposition follows now directly, because the state transition matrices have exactly the same values for the corresponding t, τ and z, ξ. (Note that for a time-invariant system stability is always uniform.)

A time-varying linear state equation is uniformly exponentially stable if there exist positive constants γ, λ such that for any t_0 and x_0 the corresponding solution satisfies

$$\parallel x(t) \parallel \leq \gamma e^{-\lambda(t-t_0)} \parallel x_0 \parallel, \quad t \geq t_0 \tag{3.9}$$

Moreover, the equation is uniformly exponentially stable if and only if there exist positive constants γ and λ such that

$$\parallel \Phi(t,\tau) \parallel \leq \gamma e^{-\lambda(t-\tau)} \tag{3.10}$$

for all t, τ such that $t \geq \tau$, (Rugh, 1993). The result in the theorem follows immediately by the equivalence of the state transition matrices.

The equivalence of the two input-output systems R_1 and R_2 can be seen directly by writing the solution

$$\begin{aligned} y(t) &= C(t)\Phi_{R_1}(t,t_0)x_0 + \int_{t_0}^{t} C(t)\Phi_{R_1}(t,\tau)B(\tau)u(\tau)d\tau + D(t)u(t) \\ &= \bar{C}e^{\bar{A}\int_{t_0}^{t} k(\nu)d\nu}x_0 + \int_{t_0}^{t} \bar{C}e^{\bar{A}\int_{\tau}^{t} k(\nu)d\nu}k(\tau)\bar{B}u(\tau)d\tau + \bar{D}u(t) \\ &= \bar{C}e^{(1/d_1)\bar{A}(z-z_0)}x_0 + \int_{z_0}^{z} \bar{C}e^{(1/d_1)\bar{A}(z-\xi)}(1/d_1)\bar{B}\bar{u}(\xi)d\xi + \bar{D}\bar{u}(z) \\ &= \bar{C}\Phi_{R_2}(z,z_0)x_0 + \int_{z_0}^{z} \bar{C}\Phi_{R_2}(z,\xi)(1/d_1)\bar{B}\bar{u}(\xi)d\xi + \bar{D}\bar{u}(z) \\ &= \bar{y}(z) \end{aligned} \tag{3.11}$$

If a finite input signal gives a finite output in R_1, the same occurs for R_2 also, and vice versa. □

By means of the weighting function a result concerning the zero state BIBO stability can easily be stated. Consider the system (3.1), in which the matrix $D(\cdot)$ is bounded. The system is zero state BIBO stable, if and only if there exists a number K such that for all $t \geq \tau$ it holds

$$\int_{\tau}^{t} \parallel p'(t,\nu) \parallel d\nu \leq K \tag{3.12}$$

For references, see e.g. (Brockett, 1970), (Jamshidi and Malek-Zavarei, 1986), (Rugh, 1993).

If the weighting function is z-invariant, the above condition can then be used with respect to z to deduce whether the system is zero state BIBO stable in z-domain. As an example, let a mixing process be described by the residence time distribution, which is assumed to be invariant with respect to z. The zero state BIBO stability is guaranteed, because the weighting function and the residence time distribution are known to be equal in z-domain, and the integral of residence time distribution from the time t_0 to infinity is 1.

Proposition 3 *If the function $z = f(t)$ that transforms R_1 into R_2 tends to infinity as $t \to \infty$, the following holds:*

1. *R_1 is controllable, if and only if R_2 is controllable.*
2. *R_1 is observable, if and only if R_2 is observable.*

Proof: The representation R_1 is controllable, if and only if for each t_0 there is a t_1 such that the *controllability Gramian*

$$W_{R_1}(t_0, t_1) = \int_{t_0}^{t_1} \Phi_{R_1}(t_0, t) B(t) B^T(t) \Phi_{R_1}^T(t_0, t)\, dt \tag{3.13}$$

is non-singular (Rugh, 1993). This can be shown to be equivalent to the fact that the rows of the matrix $\Phi_{R_1}(t_0, \cdot)B(\cdot)$ are linearly independent functions of time on $[t_0, t_1]$, (Padulo and Arbib, 1974). The condition becomes

$$U_{R_1}(t_0, \tau) = \Phi_{R_1}(t_0, \tau) k(\tau) \bar{B} = k(\tau) e^{-\bar{A} \int_{t_0}^{\tau} k(\nu) d\nu} \bar{B} \tag{3.14}$$

For R_2 the corresponding matrix is

$$\begin{aligned} U_{R_2}(z_0, \xi) &= \Phi_{R_2}(z_0, \xi)(1/d_1)\bar{B} \\ &= (1/d_1) e^{-(1/d_1)\bar{A}(\xi - z_0)} \bar{B} \end{aligned} \tag{3.15}$$

By considering (3.6) it follows easily that

$$\begin{aligned} U_{R_2}(z_0, \xi) &= (1/d_1) e^{-\bar{A} \int_{t_0}^{\tau} k(\nu) d\nu} \bar{B} \\ &= \frac{1}{d_1 k(\tau)} U_{R_1}(t_0, \tau) \end{aligned} \tag{3.16}$$

Since the term $d_1 k(\cdot)$ is always positive, the rows of $U_{R_1}(t_0, \tau)$ are linearly independent on $[t_0, \tau]$, if and only if the rows of $U_{R_2}(z_0, \xi)$ are linearly independent on $[z_0, \xi]$. That completes the proof of the first part of the theorem.

The representation R_1 is completely observable, if and only if for each t_0 there is a t_1 such that the *observability Gramian*

$$M_{R_1}(t_0, t_1) = \int_{t_0}^{t_1} \Phi_{R_1}^T(t, t_0) C^T(t) C(t) \Phi_{R_1}(t, t_0)\, dt \tag{3.17}$$

is non-singular. Again, this is equivalent to that the columns of the matrix $C(\cdot)\Phi_{R_1}(\cdot, t_0)$ are linearly independent functions of time on $[t_0, t_1]$, (Padulo and Arbib, 1974). The proof of the second part of the theorem follows by applying this result in a similar manner as above. □

Remark: Notice that the condition that z must tend to infinity as $t \to \infty$ is actually essential only in the case of asymptotic stability. When stability, controllability and observability are concerned, this presumption in the above two propositions could be relaxed.

For a simple example consider the differential equation

$$\dot{x}(t) = -e^{-t}x(t) + e^{-t}u(t)$$

with $x(t_0) = x_0$. The equation is z-invariant, and it can be written in z-domain as

$$\frac{d\bar{x}(z)}{dz} = -\bar{x}(z) + \bar{u}(z)$$

$\bar{x}(z(t_0)) = \bar{x}(0) = x_0$. The latter equation is asymptotically stable. To study the former representation, choose $u(t) \equiv 0$. The solution of the equation is then

$$x(t) = x_0 e^{-e^{-t_0}} e^{e^{-t}} \to x_0 e^{-e^{-t_0}} \neq 0$$

as $t \to \infty$. Hence, the differential equation in time domain is not asymptotically stable. The explanation for this is that the range of the z-variable

$$z(t) = \int_{t_0}^{t} e^{-\nu} d\nu = e^{-t_0} - e^{-t}$$

is $[0, e^{-t_0})$ so that the variable does not tend to infinity.

3.3 Differential models of basic mixing processes

Consider an ideally mixed vessel through which a liquid consisting of a solvent and dissolved solute is flowing continuously. An ideal mixing process but no chemical reaction is assumed to occur in the vessel. The input and output flow rates — and thus the liquid volume also — can vary, but they are assumed to be positive-valued continuous functions. With respect to concentrations the system can be modelled by the equations (Zenger, 1992)

$$\frac{d(V(t)c(t))}{dt} = Q_i(t)c_i(t) - Q_o(t)c(t) \tag{3.18}$$

$$\dot{V}(t) = Q_i(t) - Q_o(t) \tag{3.19}$$

A simple manipulation of the two equations leads to the time-varying state equation

$$\dot{c}(t) = -\frac{Q_i(t)}{V(t)}(c(t) - c_i(t)) \tag{3.20}$$

from which the weighting function is easily obtained

$$p'(t,\tau) = \frac{Q_i(\tau)}{V(\tau)} e^{-\int_\tau^t \frac{Q_i(\nu)}{V(\nu)} d\nu} \tag{3.21}$$

According to equation (2.8) the RTD is then

$$p(t,\tau) = \frac{Q_o(t)}{Q_i(\tau)} p'(t,\tau) = \frac{Q_o(t)}{V(\tau)} e^{-\int_\tau^t \frac{Q_i(\nu)}{V(\nu)} d\nu} \tag{3.22}$$

By introducing the new time scale

$$z = f(t) = \int_{t_0}^t \frac{Q_i(\nu)}{V(\nu)} d\nu \tag{3.23}$$

and by using (2.30) it follows that

$$\bar{p'}(z,\xi) = \frac{p'(t,\tau)}{\dot{f}(\tau)} = e^{-\int_\tau^t \frac{Q_i(\nu)}{V(\nu)} d\nu} = e^{-(f(t)-f(\tau))} = e^{-(z-\xi)} \tag{3.24}$$

which also fulfills (2.31) so that the system is z-invariant. For the RTD

$$\bar{p}(z,\xi) = \frac{p(t,\tau)}{\dot{f}(t)} = \frac{Q_o(t)}{Q_i(t)} \frac{V(t)}{V(\tau)} e^{-(z-\xi)} \tag{3.25}$$

which is in accordance with (2.38). Note that although the system is z-invariant, the RTD curve in the volumetric scale remains invariant if the flow rate – but not the liquid volume – is changing. If the input and output flow rates differ from each other, the RTD curve changes also.

The above results can be calculated easier by first transforming the state equation (3.20) by (3.23) into

$$\frac{d\bar{c}(z)}{dz} = -\bar{c}(z) + \bar{c}_i(z) \tag{3.26}$$

from which (3.24) follows easily.

The impulse response of the system will be calculated as an example. Let M moles of tracer be injected into the input flow at time τ. By equation (2.22) the 'strength' of the impulse is thus $N = M/Q_i(\tau)$. By using equations (2.26) and (3.21) the output concentration can be verified to be

$$c(t) = \frac{M}{V(\tau)} e^{-\int_\tau^t \frac{Q_i(\nu)}{V(\nu)} d\nu} \tag{3.27}$$

Note that at time τ the concentration is $M/V(\tau)$, which can be interpreted to be the initial concentration of the substance in the total volume of the system. This is in accordance to the definition of a perfect mixer stating that any particle entering the system can be found anywhere in the system with equal probability. It should be noted however that equation (2.22) is more profound, because it gives a relationship valid for not only perfect mixers but other flow systems as well.

Additionally, it is interesting to note that by using relationship (2.19) the output concentration can also be expressed as

$$c(t) = \frac{M}{V(t)} e^{-\int_\tau^t \frac{Q_o(\nu)}{V(\nu)} d\nu} \tag{3.28}$$

Equations (3.27) and (3.28) show implicitly that for all t the terms $\int_{-\infty}^t \frac{Q_i(\nu)}{V(\nu)} d\nu$ and $\int_t^\infty \frac{Q_o(\nu)}{V(\nu)} d\nu$ must diverge, as noticed also by Nauman (1969). It is actually a consequence of the basic assumptions (2.10) and (2.9), which can easily be shown by direct integration.

To illustrate, consider the flow rate $Q(t) = e^{-t}$, which is a positive function that approaches zero asymptotically. Let the liquid volume be a constant V. The modified time scale becomes

$$z = \frac{1}{V} \int_{t_0}^t Q(\nu) d\nu = -\frac{1}{V}(e^{-t} - e^{-t_0})$$

so that

$$z \in [0, \frac{1}{V} e^{-t_0})$$

The z-variable thus approaches a finite limiting value meaning that equations (2.9) and (2.10) do not hold. However, this confusing result is due to the very peculiar nature of the flow rate; the same example was actually considered in the end of the previous section.

The determination of the weighting function in z-domain can easily be derived in the case of a z-invariant state-space representation. Consider equation (3.1) with $x(t_0) = 0$, $A(t) = k(t)\bar{A}$, $B(t) = k(t)\bar{B}$, $C(t) = \bar{C}$, $D(t) = 0$, where $k(t)$ is a positive and continuous function. By using the transformation

$$z = f(t) = d_1 \int_{t_0}^t k(\nu) d\nu$$

$(d_1 > 0)$ the equations change into the form

$$\frac{d\bar{x}(z)}{dz} = (1/d_1)\bar{A}\bar{x}(z) + (1/d_1)\bar{B}\bar{u}(z) \tag{3.29}$$

$$\bar{y}(z) = \bar{C}\bar{x}(z) \tag{3.30}$$

with $\bar{x}(0) = 0$. The state transition matrix of the system equation is

$$\Phi(z,\xi) = e^{(1/d_1)\bar{A}(z-\xi)} \tag{3.31}$$

The impulse response or weighting function is

$$\bar{p'}(z,\xi) = (1/d_1)\bar{C}e^{(1/d_1)\bar{A}(z-\xi)}\bar{B} \tag{3.32}$$

assuming that the unit impulse $\delta_\xi(z)$ has entered at time $\xi = f(\tau)$, $\tau \geq t_0$. In time domain the result is

$$p'(t,\tau) = d_1 k(\tau)(1/d_1)\bar{C}e^{(1/d_1)\bar{A}d_1\int_\tau^t k(\nu)d\nu}\bar{B} = k(\tau)\bar{C}e^{\bar{A}\int_\tau^t k(\nu)d\nu}\bar{B} \tag{3.33}$$

Example: Consider two perfect mixers in series and assume that $V_1 = V_2 = V$. The weighting function in time-domain can easily be computed using the above equation

$$p'(t,\tau) = \frac{Q(\tau)}{V}e^{-\int_\tau^t \frac{Q(\nu)}{V}d\nu}\int_\tau^t \frac{Q(\nu)}{V}d\nu \tag{3.34}$$

It is instructive to compare the transformation technique to the use of convolution integrals as a method to determine the impulse response. The concentration at the outlet of the first vessel is

$$c_1(t_1) = \int_{t_0}^{t_1} c_0(\tau)p'(t_1,\tau)d\tau \tag{3.35}$$

where $p'(t,\tau)$ is the weighting function of one ideally mixed. Applying the convolution integral to the second vessel gives

$$c_2(t) = \int_{t_0}^{t} c_1(t_1)p'(t,t_1)dt_1 = \int_{t_0}^{t}\int_{t_0}^{t_1} c_0(\tau)p'(t_1,\tau)d\tau\, p'(t,t_1)dt_1 \tag{3.36}$$

where $t_0 \leq t_1 \leq t$. Now, let the input concentration be $c_0(t) = \delta_\tau(t)$. The expression for the output concentration simplifies to (notice the selecting property of the impulse function and the fact that $p'(t_1,\tau) = 0$, if $t_1 < \tau$)

$$c_2(t) = \int_\tau^t p'(t_1,\tau)p'(t,t_1)dt_1 \tag{3.37}$$

Substituting the expression for $p'(t,\tau)$ to the equation above gives the final result

$$c_2(t) = \int_\tau^t \frac{Q(\tau)}{V}\frac{Q(t_1)}{V}e^{-\int_\tau^t \frac{Q(\nu)}{V}d\nu}dt_1 = \frac{Q(\tau)}{V}e^{-\int_\tau^t \frac{Q(\nu)}{V}d\nu}\int_\tau^t \frac{Q(\nu)}{V}d\nu \tag{3.38}$$

Next, consider a system consisting of n perfect mixers in series. The flow rate through the mixers as well as the liquid volumes in the vessels may be time-variable. Let the output flow rates of the vessels be $Q_1(t), Q_2(t), \ldots, Q_n(t)$, and the input flow rate and input concentration of the first vessel $Q_0(t)$, $c_0(t) = c_i(t)$. Correspondingly, the output concentrations are $c_1(t), c_2(t), \ldots, c_n(t)$, and the liquid volumes in the vessels $V_1(t), V_2(t), \ldots, V_n(t)$.

The representation of one perfect mixer, equations (3.18)-(3.20), can easily be extended to the case of n mixers in series. This gives

$$\dot{c}_{k+1}(t) = -\frac{Q_k(t)}{V_{k+1}(t)}(c_{k+1}(t) - c_k(t)) \tag{3.39}$$

$$\dot{V}_{k+1}(t) = Q_k(t) - Q_{k+1}(t) \tag{3.40}$$

where $k = 0, 1, 2, \ldots, (n-1)$. The state-space representation (3.1) is thus obtained, in which $x_i(t) \stackrel{\triangle}{=} c_i(t)$, $i = 1, 2, \ldots, n$, $u(t) = c_0(t)$, $y(t) = c_n(t)$, and the matrices are

$$A(t) = \begin{bmatrix} -\frac{Q_0(t)}{V_1(t)} & 0 & 0 & 0 & \cdots & 0 \\ \frac{Q_1(t)}{V_2(t)} & -\frac{Q_1(t)}{V_2(t)} & 0 & 0 & \cdots & 0 \\ 0 & \frac{Q_2(t)}{V_3(t)} & -\frac{Q_2(t)}{V_3(t)} & 0 & \cdots & 0 \\ \vdots & & & & & \\ 0 & 0 & \cdots & 0 & \frac{Q_{n-1}(t)}{V_n(t)} & -\frac{Q_{n-1}(t)}{V_n(t)} \end{bmatrix}$$

$$B(t) = \begin{bmatrix} \frac{Q_0(t)}{V_1(t)} \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

$$C(t) = \begin{bmatrix} 0 & 0 & 0 & \cdots & 0 & 1 \end{bmatrix}$$

$$D(t) = 0$$

The representation is not z-invariant, because the relationship between flow rates and liquid volumes is complex and it is in general impossible to find a common time-variable scalar factor. Before discussing this issue further consider the special case, in which the liquid volumes are constant, but the flow rate through the system may vary. Now the representation is z-invariant with respect to

$$z = (1/V_K) \int_{t_0}^{t} Q(\nu) d\nu \tag{3.41}$$

where V_K can be any (positive) constant. If the total volume $V_K = \sum_{i=1}^{n} V_i$ is used, $Q_0(t) = Q_1(t) = \cdots = Q_n(t) \stackrel{\triangle}{=} Q(t)$, and the representation (3.3) follows with $d_1 = 1/V_K$

and

$$\bar{A} = \begin{bmatrix} -\frac{1}{V_1} & 0 & 0 & 0 & \cdots & 0 \\ \frac{1}{V_2} & -\frac{1}{V_2} & 0 & 0 & \cdots & 0 \\ 0 & \frac{1}{V_3} & -\frac{1}{V_3} & 0 & \cdots & 0 \\ \vdots & & & & & \\ 0 & 0 & \cdots & 0 & \frac{1}{V_n} & -\frac{1}{V_n} \end{bmatrix}$$

$$\bar{B} = \begin{bmatrix} \frac{1}{V_1} \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

$$\bar{C} = \begin{bmatrix} 0 & 0 & 0 & \cdots & 0 & 1 \end{bmatrix}$$

$$\bar{D} = 0$$

The impulse response can now easily be calculated from the state-space representation. As an example consider the two 'extreme' cases. If $V_1 = V_2 = \cdots = V_n$, the result is

$$\bar{p'}(z) = \frac{n^n}{(n-1)!} z^{n-1} e^{-nz} \tag{3.42}$$

If on the other hand $V_i \neq V_j$ for all $i \neq j$ it follows that

$$\bar{p'}(z) = \frac{V_K^n}{\prod_{i=1}^{n} V_i} \sum_{j=1}^{n} K_j e^{-\frac{V_K}{V_j} z} \tag{3.43}$$

in which

$$K_j = \lim_{s \to -\frac{V_K}{V_j}} \frac{1}{\prod_{\substack{i=1 \\ i \neq j}}^{n} \left(s + \frac{V_K}{V_i}\right)} = \frac{1}{\prod_{\substack{i=1 \\ i \neq j}}^{n} \left(\frac{V_K}{V_i} - \frac{V_K}{V_j}\right)} \tag{3.44}$$

From the impulse response the input-output relationship

$$\bar{c}_n(z) = \int_0^z \bar{c}_0(\nu) \bar{p'}(z - \nu) d\nu \tag{3.45}$$

follows. The same results have been obtained by Niemi (1977a), but starting from the theory of residence time distributions and using convolution integrals in time domain.

3.4 Series of perfect mixers

It is interesting to study, under which conditions the system of perfect mixers in unsteady flow and volume conditions is z-invariant. The question was originally discussed in (Zenger, 1992), and the analysis presented here is based on the original derivation.

The condition for the system to be z-invariant can be deduced from the system realization (3.39)-(3.40). The condition is

$$\frac{Q_k(t)}{V_{k+1}(t)} = l_k \frac{Q_0(t)}{V_1(t)} \tag{3.46}$$

where $k = 1, 2, \ldots, (n-1)$, and l_k is a constant for each k. The flow rates and liquid volumes in the system are assumed to be positive at every time instant. The transformation is

$$z = f(t) = \int_{t_0}^{t} \frac{Q_0(\nu)}{V_1(\nu)} d\nu \tag{3.47}$$

Applying equation (3.46) repeatedly gives

$$\frac{V_k(t)}{V_{k+1}(t)} = \frac{l_k}{l_{k-1}} \frac{Q_{k-1}(t)}{Q_k(t)} \tag{3.48}$$

where $l_0 = 1$ by definition. Unfortunately, the above formula does not give much insight on the conditions under which the system of a series of mixers becomes z-invariant. In other words, it is not easy to calculate the flow rates and volumes such that (3.46) or (3.48) hold. To analyse the system further some more assumptions are now needed. Note that

$$Q_k(t) = l_k Q_0(t) \frac{V_{k+1}(t)}{V_1(t)} \tag{3.49}$$

which makes it easy to do the reasonable assumption used also in (Guizerix, 1990) that the volume ratio of each two successive mixers is always a constant. Hence, it is assumed that

$$\frac{V_k(t)}{V_{k+1}(t)} = a_k \tag{3.50}$$

with positive constants a_k. From (3.49) it is immediately seen that the flow rates (and volumes) are then constant multiples of each other. That provides a sufficient condition for the system to be z-invariant.

To calculate any flowrate as a function of the input and output flow rates of the whole system use equations (3.40) and (3.50) repeatedly so that for any $k = 1, 2, \ldots, (n-1)$

$$\begin{aligned} Q_k(t) &= Q_{k+1}(t) + \frac{1}{a_k}\dot{V}_k(t) = Q_{k+1}(t) + \frac{1}{a_k a_{k-1}}\dot{V}_{k-1}(t) \\ &= Q_{k+1}(t) + \frac{1}{a_k a_{k-1} \cdots a_1}\dot{V}_1(t) \\ &= Q_{k+1}(t) + \frac{1}{\prod_{i=1}^{k} a_i}(Q_0(t) - Q_1(t)) \end{aligned} \tag{3.51}$$

Similar equations can be written for $Q_{k+1}(t), Q_{k+2}(t) \cdots Q_{n-1}(t)$. Substituting $Q_{k+1}(t)$ into the above equation and repeating the process for $Q_{k+2}(t)$ and so on leads to

$$Q_k(t) = Q_{k+2}(t) + \left(\frac{1}{\prod_{i=1}^{k} a_i} + \frac{1}{\prod_{i=1}^{k+1} a_i}\right)(Q_0(t) - Q_1(t)) = \cdots \tag{3.52}$$

$$= \quad Q_n(t) + (\frac{1}{\prod_{i=1}^{k} a_i} + \frac{1}{\prod_{i=1}^{k+1} a_i} + \cdots + \frac{1}{\prod_{i=1}^{n-1} a_i})(Q_0(t) - Q_1(t))$$

Let

$$S_n(k) \triangleq \frac{1}{\prod_{i=1}^{k} a_i} + \frac{1}{\prod_{i=1}^{k+1} a_i} + \cdots + \frac{1}{\prod_{i=1}^{n-1} a_i}$$

$k = 1, 2, \ldots (n-1)$, which is by definition a sum containing $n-k$ terms. The equation for $Q_k(t)$ can be written

$$Q_k(t) = Q_n(t) + S_n(k)(Q_0(t) - Q_1(t)) \tag{3.53}$$

Writing $Q_1(t)$ according to the above formula, solving for $Q_0(t) - Q_1(t)$ and substituting back into (3.53) gives

$$Q_k(t) = \frac{S_n(k)}{1 + S_n(1)} Q_0(t) + \frac{(1 + S_n(1) - S_n(k))}{1 + S_n(1)} Q_n(t) \tag{3.54}$$

It is immediately noticed that if $Q_0(t) = Q_n(t)$ then $Q_k(t) = Q_0(t) = Q_n(t)$ for all k. Subsequently, the liquid volumes would be at constant values.

The flow rates $Q_k(t)$ can be calculated, if $Q_0(t)$ and $Q_n(t)$ are continuously measured and the constants a_k are known. In the special case that the volume ratios $a_k = a$ are the same, the following simplified expression for $S_n(k)$ is obtained

$$S_n(k) = \begin{cases} \frac{1 - (\frac{1}{a})^{(n-k)}}{a^{k-1}(a-1)} & a \neq 1 \\ n - k & a = 1 \end{cases} \tag{3.55}$$

The condition $a_k = a = 1$ means that the liquid volumes in the different vessels are the same, although they are varying as a function of time. In that case equation (3.54) simplifies to

$$Q_k(t) = \frac{1}{n}((n-k)Q_0(t) + kQ_n(t)) \tag{3.56}$$

which has also been reported in (Guizerix, 1990).

If the system is assumed to be z-invariant, equations (3.49) and (3.50) give

$$Q_k(t) = \frac{l_k}{a_1 a_2 \cdots a_k} Q_0(t) \tag{3.57}$$

Solving for l_k and then using (3.54) gives the values for the constants l_k, $k = 1, 2, \ldots, (n-1)$.

$$l_k = a_1 a_2 \cdots a_k (\frac{S_n(k)}{1 + S_n(1)} + \frac{(1 + S_n(1) - S_n(k))}{(1 + S_n(1))} \frac{Q_n(t)}{Q_0(t)}) \tag{3.58}$$

Because the numbers l_k are assumed to be constant, the ratio between the output flow rate and the input flow rate $(Q_n(t)/Q_0(t))$ must also be constant. Note that if $Q_n(t) \neq Q_0(t)$, the vessels will eventually become empty or there will be an overflow.

According to the definition of $S_n(k)$ it holds that

- $S_n(k) > 0$ for all k.
- $S_n(i) > S_n(j)$, if and only if $i < j$, so that $1 + S_n(1) - S_n(k) \geq 1$ for all k.

and further

$$l_k > a_1 a_2 \cdots a_k \frac{S_n(k)}{1 + S_n(1)} \tag{3.59}$$

which gives the lower limits for the constants l_k.

The result obtained can be stated again in a compact form. Given that the volume ratios of the vessels are constants a_k (according to equation (3.50)), the representation of the system is z-invariant, if and only if the ratio between the output flow rate and the input flow rate is constant. The flow rates between the intermediate vessels are then given by (3.57), where the values l_k are obtained from (3.58).

If the output flow rate and input flow rate are equal, then $l_k = a_1 a_2 \cdots a_k$ so that $Q_k(t) = Q_0(t)$ as noted earlier. If the volumes of the vessels are equal ($a_k = 1$ for all k), the expression for l_k simplifies to

$$l_k = \frac{n-k}{n} + \frac{k}{n}\frac{Q_n(t)}{Q_0(t)} \tag{3.60}$$

In spite of the rather extensive analysis presented above, it should be remembered that the conditions under which a series of mixers with variable flow and volume is z-invariant are restrictive. The complexity of a system with several process units under variable flow and volume can generally not be reduced by the use of a modified time scale.

Under the assumption of constant volume ratios, the dynamic behaviour of the concentrations can be calculated by determining $Q_k(t)$ continuously and applying repeatedly the formula

$$c_{k+1}(t) = \int_{t_0}^{t} c_k(\nu) p'(t,\nu) d\nu = \int_{t_0}^{t} c_k(\nu) \frac{Q_k(\nu)}{V_{k+1}(\nu)} e^{-\int_\nu^t \frac{Q_k(\tau)}{V_{k+1}(\tau)} d\tau} d\nu \tag{3.61}$$

with $k = 0, 1, 2, \ldots, (n-1)$.

Chapter 4

Systems With Time Delays

In Chapter 3 it was seen that time-varying linear differential systems are a natural way to model mixing processes under unsteady flow and volume. By considering a system model consisting of a combination of perfect mixers and recycle and bypass flows a large spectrum of typical flow systems can be covered (Zenger, 1992). But nothing has so far been said about delays, which are commonly present in process control applications. In flow processes the delay can in a natural way be modelled by a *plug flow vessel*, through which the process material is assumed to flow without any mixing occurring. The concentration of the solute at the outlet of the vessel is the same as in the inlet a certain time ago. Under steady flow conditions the delay time can be calculated by dividing the liquid volume in the vessel by the flow rate.

In this chapter the delay is studied under the assumption of unsteady flow conditions. The delay caused by the plug flow vessel turns out to be time-varying, and the *delay function* is introduced to model it. It is shown that a similar change of time scale as in Chapter 3 can be used to change the delay into a constant. The result gives the tools to study a time-variable flow system consisting of perfect mixers, plug flow vessels and recycling by first transforming the model into a form with constant coefficients and constant delays. The delay function has some interesting properties, which are discussed in detail.

4.1 The transformation of delays

Consider a delay system with input $u(t)$ and output $y(t)$, which are related by

$$y(t) = u(t - T_d(t)) \tag{4.1}$$

where $T_d(\cdot)$ is a non-negative continuous function (the delay function). The equation means that the output at time t is the same as the input at time $t - T_d(t)$. For example, a plug flow vessel with a constant flow rate can be modelled by

$$c(t) = c_i(t - V/Q) \tag{4.2}$$

where $c(\cdot)$ and $c_i(\cdot)$ denote output and input concentrations, respectively. V is liquid volume in the vessel and Q the flow rate through the vessel. The constant delay (function) is obtained by dividing the liquid volume in the vessel by the flow rate. In the case of varying flow rates and liquid volumes the determination of the delay function is not so straightforward as it will be seen in what follows.

Let $z = f(t)$, where $f : t \mapsto f(t)$ is a positive, continuously differentiable and monotonously increasing function with the inverse $h : z \mapsto h(z)$. Use the same abbreviations as before viz. $\bar{u}(z) = u(h(z))$, $\bar{y}(z) = y(h(z))$. If equation (4.1) can be written in the z-domain as

$$\bar{y}(z) = \bar{u}(z - z_c) \tag{4.3}$$

where z_c is a constant, it is called z-invariant. The necessary and sufficient condition for equation (4.1) to be z-invariant is given in the following proposition.

Proposition 4 *Let system* (4.1) *be defined on a time interval T. The equation is z-invariant, i.e.* (4.3) *holds, if and only if there exists a positive constant z_c such that for all t*

$$f(t) - f(t - T_d(t)) = z_c \tag{4.4}$$

Proof: Suppose first that

$$f(t) - f(t - T_d(t)) = z_c$$

holds. Then

$$t - T_d(t) = h(f(t - T_d(t))) = h(f(t) - z_c) = h(z - z_c)$$

so that

$$y(t) = y(h(z)) = u(t - T_d(t)) = u(h(z - z_c))$$

which is the same as

$$\bar{y}(z) = \bar{u}(z - z_c)$$

Conversely, suppose that

$$\bar{y}(z) = \bar{u}(z - z_c)$$

or

$$y(h(z)) = u(h(z - z_c))$$

But according to (4.1) this is the same as

$$y(t) = u(t - T_d(t))$$

which means

$$t - T_d(t) = h(z - z_c)$$

Hence

$$f(t - T_d(t)) = z - z_c = f(t) - z_c$$

or

$$f(t) - f(t - T_d(t)) = z_c$$

□

Remark: In the above proposition the system model is assumed to be valid on a specified time interval. If $T_0 = [t_0, \infty)$, then $T = [t_1, \infty)$, where $t_1 > t_0$ is chosen such that for all $t \geq t_1$, $t - T_d(t) \in T_0$ and $y(t) = 0$ for all $t < t_1$.

As mentioned earlier, under stationary conditions the input-output relationship of the plug flow vessel is given by (4.2). If the flow rate changes, i.e. $Q = Q(t)$, one might be tempted to write

$$c(t) = c_i(t - V/Q(t))$$

which is however incorrect. This is easy to see by considering a concentration pulse (or particle) at the output of the vessel at time t. The time that has elapsed since the pulse entered the vessel is not generally given by $V/Q(t)$, because the flow rate may have been continuously changing during the time that the pulse has stayed in the vessel.

The correct form of the equation is obtained as follows. During the time that a hypothetical concentration pulse stays in the vessel, the volume V of material must pass through the vessel irrespective of flow changes. This can be stated mathematically by

$$\int_{t-T_d(t)}^{t} (Q(\tau)/V)d\tau = 1 \tag{4.5}$$

which holds for all t on the time interval T. The equation can also be regarded as the definition of the *delay function* $T_d(\cdot)$. An important result follows, when a change of variables is done

$$z = f(t) = (1/V)\int_{t_0}^{t} Q(\tau)d\tau \tag{4.6}$$

so that

$$f(t) - f(t - T_d(t)) = \int_{t-T_d(t)}^{t} (Q(\tau)/V)d\tau = 1 \tag{4.7}$$

By Proposition 4 the model for the plug flow vessel in z-domain is then

$$\bar{c}(z) = \bar{c}_i(z - 1) \tag{4.8}$$

and the time-varying nature of the equation has disappeared. Note that z_c has the value 1, if the scaling factor V in the formula for the z-scale is the constant liquid volume in the plug flow vessel. If another coefficient is used, z_c will have some other constant value. Under steady operation conditions ($Q(t) = Q$, constant) the previous calculations give $(Q/V)T_d(t) = z_c = 1$, so that $T_d(t) = V/Q$ as expected. In the time-varying case the determination of the delay function is generally a difficult problem, as might be expected from equation (4.5).

The above result (4.8) can also be derived by using a different approach. Let the number of equal-sized perfect mixers in series tend to infinity so that the total liquid volume in the system ($V_K = nV$) is constant. Taking the Laplace transformation with respect to z in equation (3.42) gives

$$\bar{P}'_n(s) = \frac{n^n}{(s+n)^n} = \frac{1}{(1+\frac{s}{n})^n} \tag{4.9}$$

Letting n approach infinity leads to

$$\lim_{n\to\infty} \bar{P}'_n(s) = e^{-s} \tag{4.10}$$

Hence it holds that

$$\lim_{n\to\infty} \bar{p}'_n(z,0) = \delta(z-1) \tag{4.11}$$

which denotes the same input-output behaviour as (4.8). The result confirms the well-known fact that a growing number of equal perfect mixers in series approximates the dynamics of a plug flow vessel, if the total liquid volume is kept constant.

The delay function has some interesting theoretical properties, which are studied in the next section.

4.2 Properties of the delay function

Consider a plug flow vessel with varying flow rate but constant liquid volume. The delay function $T_d(t)$ is defined by (4.5). If t_0 is the absolute time origin, it is assumed that $t - T_d(t) \geq t_0$ for all t, so that the signals are properly defined. Moreover, $T_d(\cdot)$ is assumed to be a positive function. It can be shown (Zenger, 1994) that a function having these properties and fulfilling (4.5) is continuously differentiable.

By differentiating (4.5) and solving for $\dot{T}_d$ gives

$$Q(t) - Q(t - T_d(t))(1 - \dot{T}_d(t)) = 0 \tag{4.12}$$

and further

$$\dot{T}_d(t) = 1 - \frac{Q(t)}{Q(t - T_d(t))} \tag{4.13}$$

which makes it possible to simulate the delay function numerically. The initial condition can be obtained e.g. by assuming that the flow rate has been constant during the initial time period and $T_d(t) = V/Q$ holds initially.

Based on the Proposition 4 it is easy to provide a numerical way to calculate the delay function. The algorithm is formulated in discrete time so that the delay function can be calculated on line as flow measurements are obtained from the process. Furthermore, no initial conditions are needed. The algorithm presumes that the time values and the corresponding values of the z-variable are stored so that the values for $t - T_d(t)$ exist for all t. Assume that the constant delay of the plug flow vessel is z_c, if the modified time scale $f : t \mapsto f(t)$ is used. The algorithm can be given as follows:

Algorithm:

- Measure the flow rate continuously at discrete time instants.
- Store the values of t and $z = f(t)$ in each time instant.
- For every t, calculate $z_1 = f(t) - z_c$ and determine the nearest time instant t_1 that corresponds to the value z_1.
- Determine the value of the delay function at time t as $T_d(t) \approx t - t_1$.
- Delete the stored values related to smaller time instants than t_1.

The algorithm is a straightforward application of the definition of the delay function and of the fact that the delay is constant in z-domain. Naturally, the accuracy depends on the measurement interval, i.e. the discretization interval of the delay function. The 'nearest' time instant corresponding to z_1 can be calculated by interpolation or simply by choosing the time value corresponding to the stored value z that has the largest value smaller than or equal to z_1. The delay function is considered to be undefined until enough data is available to include values corresponding to z_1.

The delay function has some interesting properties, which are not immediately obvious. 'Definition' (4.5) is a mathematical way to describe the time that a pulse stays in the vessel. In order for the solution function to be physically meaningful the following points must be noticed.

- If the delay function is defined on an interval T, then for each $t \in T$ the function must satisfy $t - T_d(t) \in T_0$, where T_0 is the time interval in which the system is assumed to be in operation.
- $T_d(t)$ is a positive and continuously differentiable function.

- For all $t \in T$ it holds $\dot{T}_d(t) < 1$.

As mentioned in Section 4.1, if $Q(\cdot)$ is defined on the interval $[t_0, \infty)$, the definition interval of the delay function can be chosen as $T = [t_1, \infty)$, in which t_1 is such that equation (4.5) is properly defined. For the second item, note that in Proposition 4 only the continuity of the delay function was assumed; however the 'physical' relationship (4.5) implies that the function is even continuously differentiable. The differentiability also makes the derivation of (4.13) from (4.5) justified. From the differential equation of the delay function the last condition in the above list is immediately obvious, because the flow rate has been defined to be positive. It is interesting, though, that this property can be explained also by direct physical properties of the plug flow vessel. To this end, consider the function

$$\eta(t) = t - T_d(t) \tag{4.14}$$

which states that a particle leaving the vessel at time t has entered at time $\eta(t)$. The condition $\dot{T}_d(t) < 1$ is equivalent to

$$\dot{\eta}(t) > 0 \tag{4.15}$$

Suppose that the previous equation would not hold, i.e. $\dot{\eta}(t)$ could be negative or zero at some time instant(s). That would mean that a particle at the outlet of the vessel at time $t + dt$ would have entered the vessel earlier than or at the same time as another particle, which leaves the vessel at time t. But in a plug flow vessel this is impossible so that necessarily $\dot{T}_d(t) < 1$ (or $\dot{\eta}(t) > 0$).

The function $\eta(t)$ gives the absolute time instant at which a particle leaving the vessel at time t has entered. The delay function on the other hand gives the total time that the particle stays in the vessel. It is also possible to define a function, which indicates the departure time of a particle that has entered at time t, (Nihtilä, 1991). To this end, note that the derivative of $\eta(\cdot)$ is positive; therefore it has a unique and monotonically increasing inverse function $r(\cdot)$ such that $\eta(r(t)) = t$. Combining this with (4.14) leads to

$$\eta(r(t)) = r(t) - T_d(r(t)) \tag{4.16}$$

so that

$$r(t) - T_d(r(t)) = t \tag{4.17}$$

If a particle enters the vessel at time t, it leaves at time $r(t)$ and spends the time $T_d(r(t))$ in the vessel. Taking the derivative of the previous equation gives

$$\dot{r}(t) - \dot{T}_d(r(t))\dot{r}(t) = 1 \tag{4.18}$$

so that

$$\dot{r}(t) = \frac{1}{1 - \dot{T}_d(r(t))} = \frac{Q(t)}{Q(r(t))} \tag{4.19}$$

where equation (4.13) has been used. The same result can also be derived by writing the following condition to the plug flow vessel (c.f. with (4.5))

$$\int_t^{r(t)} \frac{Q(\nu)}{V} d\nu = 1 \tag{4.20}$$

By taking the derivative and solving for $\dot{r}(t)$ gives (4.19).

Note that equation (4.19) cannot usually be solved at time t, because $r(t)$ is a time instant in the future. However, in (Nihtilä, 1991) the concept of the function r has been used in the design of control algorithms for systems that have time-varying input delays.

Next, consider the "natural" approximation of the delay function

$$T_{dap}(t) = \frac{V}{Q(t)} \tag{4.21}$$

Its derivative is then

$$\dot{T}_{dap}(t) = -V \frac{\dot{Q}(t)}{(Q(t))^2} \tag{4.22}$$

which shows that the slope of the approximation can exceed the value 1, if the flow rate decreases rapidly. At least in that case the approximation can be expected to give bad results regarding the delay function.

It is also interesting to study the error caused by incorrect initial value when applying equation (4.13). Consider a nominal solution $T_d^*(t)$ starting from a given initial segment during the time interval $[t_0, t_1]$. Then consider another solution $T_d(t)$ such that $T_d^*(t_1)$ and $T_d(t_1)$ differ by a small amount. The function $\xi(t) = T_d(t) - T_d^*(t)$ describes the error of the solution with respect to a difference in the initial condition. The main interest is to consider the stability of $\xi(\cdot)$; if the function is asymptotically stable in the sense that it remains bounded and converges to zero, the motion $T_d(\cdot)$ is also stable meaning that the error caused by an incorrect initial value is "small" and vanishes as time goes to infinity.

Note that the above idea of the stability of a trajectory with respect to a nominal trajectory is a natural extension to the classical stability theory, in which the stability of an equilibrium point has usually been considered. For more on that issue, see e.g. (Willems, 1970) or (Mohler, 1991).

By writing equation (4.13) as

$$\dot{T}_d(t) = u(T_d(t), t) \tag{4.23}$$

the derivative of $\xi(t) = T_d(t) - T_d^*(t)$ becomes

$$\dot{\xi}(t) = u(\xi(t) + T_d^*(t), t) - u(T_d^*(t), t) \stackrel{\triangle}{=} u_1(\xi(t), t) \tag{4.24}$$

which satisfies $u_1(0,t) = 0$. The stability of the motion $T_d(\cdot)$ is then equivalent to the stability of the differential equation

$$\dot{\xi}(t) = u_1(\xi(t), t) \tag{4.25}$$

with respect to the equilibrium state $\xi(0) = 0$. In terms of (4.13) the differential equation becomes

$$\dot{\xi}(t) = \frac{Q(t)}{Q(t - T_d^*(t))} - \frac{Q(t)}{Q(t - T_d^*(t) - \xi(t))} \tag{4.26}$$

The corresponding linearized equation around $\xi = 0$ is

$$\delta\dot{\xi}(t) = -\frac{Q(t)\dot{Q}(t - T_d^*(t))}{(Q(t - T_d^*(t)))^2}\delta\xi(t) \tag{4.27}$$

from which the stability properties of the perturbed solution around the nominal one can be investigated. It is obvious that at least decreasing flow rates can cause stability problems meaning that the error in the simulated delay function (according to (4.13)) increases, if the initial value is incorrect.

4.3 Systems containing perfect mixers and plug flow vessels

In Chapter 3 it was shown that under variable flow but constant liquid volume a series of perfect mixers is z-invariant. In the current chapter the same was found to hold for one plug flow vessel. It is natural to consider whether similar conclusions can be made concerning systems consisting of an arbitrary topology of perfect mixers, plug flow vessels and possible recycling or bypass flows. Results of this nature are particularly important, because it is common practice to use such models in the description of the dynamics of real processes, see e.g. (Thereska *et al.*, 1996), (Bazin and Hodouin, 1988). The idea is then to model a process by e.g. a tracer test to determine the RTD, after which a structural model is used and the parameters of it are fitted to match the measured RTD as closely as possible.

It turns out that the previous results can be extended to a composition of basic unit processes relatively easy. To this end, consider the representation

$$\dot{x}(t) = A(t)x(t) + B(t)u(t) + \sum_{i=i_1}^{i_d} a_i(t)x_i(t - T_{dxxi}(t)) + \sum_{j=j_1}^{j_d} b_j(t)u_j(t - T_{dxuj}(t)) \tag{4.28}$$

$$y(t) = C(t)x(t) + D(t)u(t) + \sum_{k=k_1}^{k_d} c_k(t)x_k(t - T_{dyxk}(t)) + \sum_{l=l_1}^{l_d} d_l(t)u_l(t - T_{dyul}(t)) \tag{4.29}$$

in which $x(\cdot) \in (\Re^n)^\Re$, $u(\cdot) \in (\Re^m)^\Re$, $y(\cdot) \in (\Re^r)^\Re$, $A(\cdot) \in (\Re^{n\times n})^\Re$, $B(\cdot) \in (\Re^{n\times m})^\Re$, $C(\cdot) \in (\Re^{r\times n})^\Re$, $D(\cdot) \in (\Re^{r\times m})^\Re$ are continuous functions. The scalar functions $x_i(\cdot)$, $x_k(\cdot)$, $u_j(\cdot)$, $u_l(\cdot)$ consist of those terms in the state and input variables that contain delays. The delays are assumed to be positive and differentiable functions. The coefficients $a_i(\cdot)$, $b_j(\cdot)$, $c_k(\cdot)$, $d_l(\cdot)$ are scalar-valued continuous functions.

Consider equations (4.28) and (4.29) on a time interval T. The representation is called *z-invariant*, if the change of the time variable with another variable z leads to state equations with constant coefficients and constant delay terms, and where the input-, state- and output variables of the original and new representation have a one-to-one relationship. The following theorem gives necessary and sufficient conditions for the representation to be z-invariant.

Proposition 5 *Let the system be described by equations (4.28) and (4.29). The representation is z-invariant, if and only if the following two conditions hold.*

1. *For the coefficient matrices and scalar functions it holds that $A(t) = k(t)\bar{A}$, $B(t) = k(t)\bar{B}$, $C(t) = \bar{C}$,$D(t) = \bar{D}$, $a_i(t) = k(t)\bar{a}_i$, $b_j(t) = k(t)\bar{b}_j$, where $k(t)$ is a positive and continuous scalar-valued function, $\bar{A}$, $\bar{B}$, $\bar{C}$, $\bar{D}$ are constant matrices, and $\bar{a}_i$,$\bar{b}_j$, $c_k(t) = \bar{c}_k$, $d_l(t) = \bar{d}_l$ are constants for each index value.*

2. *Every function $z = f(t) = d_1 \int_{t_0}^{t} k(\tau)d\tau + d_2$ with $d_1 > 0, d_2 \geq 0$, fulfils the equations*

$$f(t) - f(t - T_{dxxi}(t)) = z_{xxi}$$

$$f(t) - f(t - T_{dxuj}(t)) = z_{xuj}$$

$$f(t) - f(t - T_{dyxk}(t)) = z_{yxk}$$

$$f(t) - f(t - T_{dyul}(t)) = z_{yul}$$

   where z_{xxi}, z_{xuj}, z_{yxk}, z_{yul} are constants for every index i, j, k and l.

The representation in z-domain is then

$$\frac{d\bar{x}(z)}{dz} = (1/d_1)\bar{A}\bar{x}(z) + (1/d_1)\bar{B}\bar{u}(z) + (1/d_1)\sum_{i=i_1}^{i_d} \bar{a}_i\bar{x}_i(z - z_{xi}) + (1/d_1)\sum_{j=j_1}^{j_d} \bar{b}_j\bar{u}_j(z - z_{xj})$$

$$\bar{y}(z) = \bar{C}\bar{x}(z) + \bar{D}\bar{u}(z) + \sum_{k=k_1}^{k_d} \bar{c}_k\bar{x}_k(z - z_{yk}) + \sum_{l=l_1}^{l_d} \bar{d}_l\bar{u}_l(z - z_{yl})$$

The proof is simply a technical combination of Propositions 1 and 4, and is therefore omitted here.

The significance of the above proposition becomes evident by noting that in the case of variable flow but constant volume all models containing ideally mixed vessels, plug flow vessels and bypass (recycle) flows with constant flow ratios can be transformed into z-domain to get a representation with constant coefficient matrices and constant delay terms (Zenger, 1992).

4.4 Examples

To give some idea on the explicit expressions of delay functions, a few examples are now considered. Assume a plug flow vessel with a constant liquid volume V, and let the flow rate through the vessel be $Q(t) = 1/t$, $t > 0$. By using the equation (4.5) and taking the condition $t - T_d(t) > 0$ into account, the delay function is found to be

$$T_d(t) = t(1 - e^{-V}) \tag{4.30}$$

which is a straight line. The differential equation (4.13) is

$$\dot{T}_d(t) = \frac{T_d(t)}{t} \tag{4.31}$$

which is satisfied by the above delay function. The solution to the differential equation is $T_d(t) = (T_d(t_0)/t_0)t$, which shows that an incorrect initial value at some time instant $t_0 > 0$ (e.g. in simulation) leads to a constantly increasing absolute error in the delay function. The result is in accordance with the theoretical analysis in Section 4.2. If $T_0 = [t_0, \infty)$, $t_0 > 0$, the correct initial value in the simulation can be taken as $t_0(e^V - 1)$ at time $t_1 = t_0 e^V$.

In the example the initial value determines the slope of the simulated delay function. It should be noticed that differential equation (4.13) is also satisfied by the incorrect candidates of the delay function v.i.z. $t(1 + e^{-V})$, the other solution of (4.5) for which $t - T_d(t) < 0$, and $V/Q(t) = Vt$, the approximation (4.21). The function $t(1 + e^{-V})$ has a slope larger than 1, so that it is clearly incorrect. The function $V/Q(t) = Vt$ would have the correct slope, if $1 - e^{-V} = V$, but that holds only if $V = 0$.

Consider next the rather 'pathological' case of exponentially decreasing flow rate $Q(t) = e^{-t}$, which is defined for all $t \in (-\infty, \infty)$. An analytical solution of the delay function can easily be shown to be

$$T_d(t) = ln(Ve^t + 1) = ln(\frac{V}{Q(t)} + 1) \tag{4.32}$$

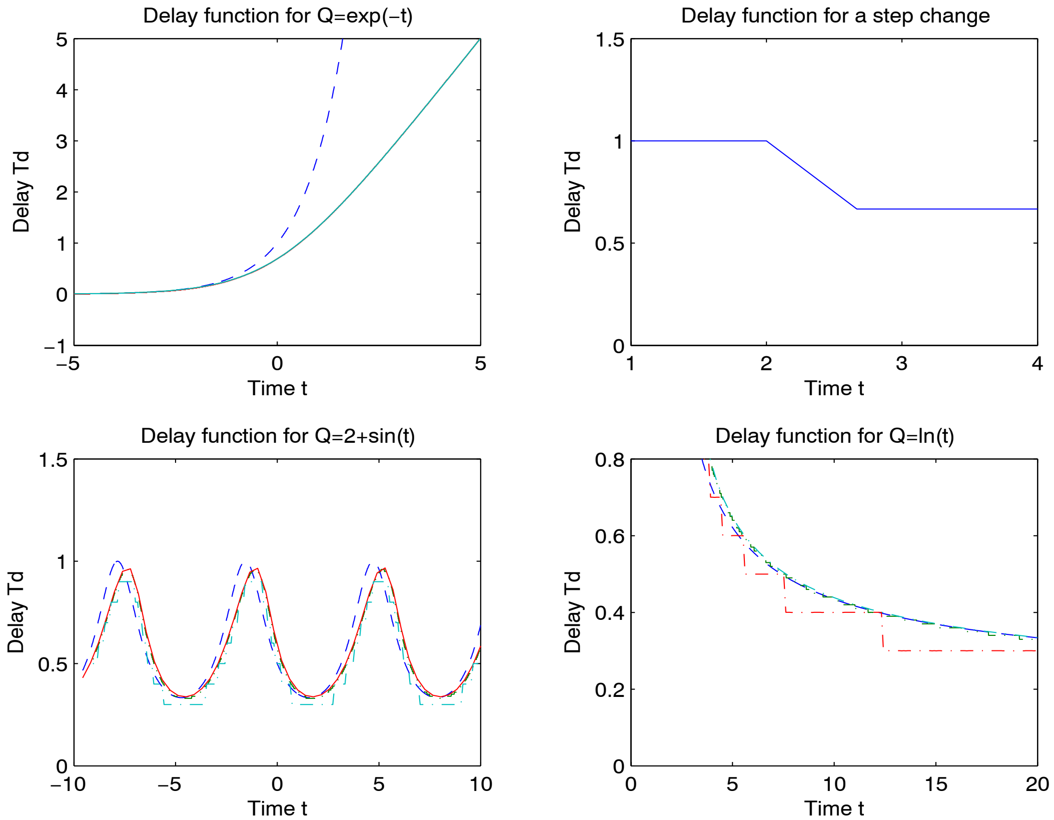

Figure 4.1: Delay functions related to different flow rates

For small time values the flow rate is large, and therefore the delay time is very small. As the flow rate decreases exponentially, the delay function grows without a limit. The approximation of the delay $V/Q(t) = Ve^t$ can give highly erroneous results, because for $t > -ln(V)$ the slope exceeds the value 1. Note, however, that for large values of time the exact solution behaves almost in a linear manner. This is because

$$T_d(t) \approx ln(Ve^t) = ln(V) + t \tag{4.33}$$

The differential equation of the delay function is correspondingly

$$\dot{T}_d(t) = 1 - e^{-T_d(t)} \tag{4.34}$$

If $T = [t_0, \infty)$, where $-\infty < t_0 < \infty$, the initial value is $ln(e^{-t_0}/(e^{-t_0} - V))$ at time $t_1 = ln(1/(e^{-t_0} - V))$. From that it can be deduced that the constant liquid volume in the vessel is restricted to the values $V < e^{-t_0}$. This is an interesting result, which can be explained by the rapidly decreasing flow rate. If the liquid volume in the vessel is too large, a particle entering the vessel at time t_0 will never reach the outlet. Another reflection of this phenomenon follows by looking at the modified time scale (4.6). The value of z is in this case restricted to the interval $[0, \frac{1}{V}e^{-t_0})$, which means that the modified time will never grow beyond a limit value. Additionally, if $V \geq e^{-t_0}$ the modified time z will not reach the value 1 in a finite time meaning that the total flow through the system (starting at time t_0) will never exceed the value V.

The exact solution of (4.34) is

$$T_d^*(t) = ln(1 + k^* e^{t-t_0}) \tag{4.35}$$

in which $k^* = e^{T_d^*(t_0)} - 1$. If an erroneous initial value is used, the absolute error becomes

$$T_d^*(t) - T_d(t) = ln(\frac{1 + k^* e^{t-t_0}}{1 + k e^{t-t_0}}) \approx ln\frac{k^*}{k} \tag{4.36}$$

where the approximation is good for large values of t. The error approaches a constant value, which depends on the error of the initial value.

The delay function is shown in Fig. 4.1 (upper left picture) for $V = 1$. The solid line represents the accurate solution, whereas the dashed line gives the approximation $V/Q(t)$. It is noticed that for small time values the fraction $V/Q(t)$ is small, and the approximation coincides well with the actual delay function. For larger time values the approximative solution gets very poor.

The solid line in the figure actually shows three results, which are on top of each other. They have been obtained by using the explicit solution of the delay function, by simulation based on the differential equation, and by using the numerical algorithm with the sampling interval 0.01. The results cannot be distinguished from each other.

Notice, however, that in this example the flow rate represents an extreme case. (The same example was actually discussed in another context in the end of Section 3.2.) In what follows it will be demonstrated that if the flow rate through the process changes within reasonable limits only, the approximative solution $V/Q(t)$ usually gives pretty good results.

Consider the case, in which an abrupt change in the flow rate from one stationary value to another occurs. If the constant flow rate $Q(t) = Q_0$ changes at time t_0 to the value $Q(t) = kQ_0$, where k is a positive constant, the delay changes from one constant value V/Q_0 dynamically according to equation (4.13)

$$\dot{T}_d(t) = 1 - k \tag{4.37}$$

The slope of the delay function is a constant until the time instant $t_1 = t_0 + (V/(kQ_0))$, after which the delay remains constant, $T_d(t) = V/(kQ_0)$. The delay, which has been obtained by using the numerical values $Q_0 = 1$, $V = 1$, $k = 1.5$, $t_0 = 2$, is shown in Fig. 4.1 (upper right figure).

Note that the flow rate in this example is not a continuous function so that $T_d(\cdot)$ is not differentiable at each time point. That does not lead to theoretical difficulties, see Remark 2 on page 20.

Next, consider a sinusoidally varying flow rate $Q(t) = 2 + sin(t)$. The application of equation (4.5) leads to

$$2T_d(t) + cos(t - T_d(t)) = V + cos(t) \tag{4.38}$$

which is hard to solve analytically. The differential equation according to (4.13) is

$$\dot{T}_d(t) = \frac{sin(t - T_d(t)) - sin(t)}{2 + sin(t - T_d(t))} \tag{4.39}$$

In Fig. 4.1 (lower left picture) the delay function has been calculated by three methods. The dashed line shows the approximation $V/Q(t)$, where $V = 1$. For $t_0 = -10$ the initial time for the simulation based on the differential equation has been calculated to be $t_1 \approx -9.57$, so that $T_d(-9.57) \approx -9.57 + 10 = 0.43$. The solid line shows the result of the simulation. The dashed-dotted curves show the approximations obtained by using the numerical algorithm with discretization intervals 0.01 and 0.1. In the former case the result cannot be distinguished from the simulated solution; however, for the larger interval 0.1 the numerical error can already be seen (the curve with clearly noticeable rectangular corner points).

It is seen that in this case the easiest way to approximate the delay function, $T_d(t) \approx V/Q(t)$, is reasonably good and can be expected to be accurate enough for most applications.

Almost identical conclusions can be made from the case $Q(t) = ln(t)$, see Fig. 4.1 (lower right picture). The dashed and solid lines show the approximative and simulated solutions, respectively, which are almost the same. The dashed-dotted curves show the results obtained by the numerical algorithm. The values are $V = 1$, $t_0 = 1.1$, $t_1 \approx 2.72$, $T_d(2.72) \approx 2.72 - 1.1 = 1.62$, $dt = 0.1, 0.01$. The largest discretization interval gives the result with a noticeable deviation from the other curves.

As the last example consider a system consisting of an ideally mixed vessel and a plug flow vessel in series such that the output flow of the perfect mixer is led through the plug flow vessel. The flow rate through the vessels, $Q(t)$, is assumed to be time-varying. The constant liquid volumes in the vessels are V_1 and V_2, respectively.

The state equations of the system are

$$\begin{aligned} \dot{c}_1(t) &= -(Q(t)/V_1)c_1(t) + (Q(t)/V_1)c_0(t) \\ y(t) &= c_1(t - T_d(t)) \end{aligned} \tag{4.40}$$

where $c_0(t)$ and $c_1(t)$ are the input and output concentrations of the perfect mixer, and $y(t) = c(t)$ is the output concentration of the plug flow vessel. Consider the modified time scale

$$z = f(t) = \frac{1}{(V_1 + V_2)} \int_0^t Q(\nu) d\nu \tag{4.41}$$

It follows that

$$\begin{aligned} f(t) - f(t - T_d(t)) &= \frac{1}{(V_1 + V_2)} \int_{t-T_d(t)}^t Q(\nu) d\nu \\ &= \frac{V_2}{(V_1 + V_2)} \int_{t-T_d(t)}^t (Q(\nu)/V_2) d\nu = \frac{V_2}{V_1 + V_2} = z_c \end{aligned} \tag{4.42}$$

where the property corresponding to equation (4.5) for a plug flow vessel has been taken into account. The representation of the system is z-invariant

$$\begin{aligned} \frac{d\bar{c}_1(z)}{dz} &= -\frac{(V_1 + V_2)}{V_1} \bar{c}_1(z) + \frac{(V_1 + V_2)}{V_1} \bar{c}_0(z) \\ \bar{y}(z) &= \bar{c}_1\left(z - \frac{V_2}{V_1 + V_2}\right) \end{aligned} \tag{4.43}$$

The behaviour of the system can now be analyzed e.g. by taking the Laplace-transformation to obtain the transfer function. The output can be calculated by using the input function (in z-domain), calculating the output and transforming back into time domain.

An interesting question arises from the previous analysis. How will the situation change, if the two vessels are placed in reverse order i.e. the output flow from the plug flow vessel is led into the perfect mixer? In the stationary case the two systems are input-output equivalent, but if the flow rate is allowed to vary, the situation is not so clear anymore. The state equations in that case are

$$\begin{aligned} \dot{c}(t) &= -(Q(t)/V_1) c(t) + (Q(t)/V_1) c_0(t - T_d(t)) \\ y(t) &= c(t) \end{aligned} \tag{4.44}$$

in which $c_o(t)$ is the input concentration of the plug flow vessel, and the state variable $c(t)$ is the output concentration of the perfect mixer.

Using the same modified time scale as above the equations become

$$\begin{aligned} \frac{d\bar{c}(z)}{dz} &= -\frac{(V_1 + V_2)}{V_1} \bar{c}(z) + \frac{(V_1 + V_2)}{V_1} \bar{c}_0\left(z - \frac{V_2}{V_1 + V_2}\right) \\ \bar{y}(z) &= \bar{c}(z) \end{aligned} \tag{4.45}$$

The input-output behaviour of the two system configurations turns out to be the same. That result follows easily by calculating the transfer functions of (4.43) and (4.45), which are the same. Actually, the two systems are *zero-state equivalent*, which can also be proved directly from the representations (4.40) and (4.44).

The result of the above example is interesting, but it should be pointed out that it represents an example case only. For time-varying system models the order of the system 'blocks' is usually important.

4.5 The case with variable volume

The analysis presented in Section 3.4 showed that the models with varying liquid volumes are more complex than those with varying flow rates only, and it is often impossible to find a transformation that would change the representation into one with constant coefficients. Consider the case of a plug flow vessel, in which both the throughput flow rate and the liquid volume change. The model equations are

$$y(t) = u(t - T_d(t)) \tag{4.46}$$

$$\dot{V}(t) = Q_i(t) - Q_o(t) \tag{4.47}$$

$$\int_{t-T_d(t)}^{t} Q_i(\tau)d\tau = V(t) \tag{4.48}$$

in which $u(\cdot)$ and $y(\cdot)$ are the input and output concentrations, $Q_i(\cdot)$ and $Q_o(\cdot)$ are the flow rates, and $V(\cdot)$ is the liquid volume in the plug flow vessel. Equation (4.48) can again be understood as the definition of the delay function. It is the property of an (ideal) plug flow vessel that during the time that a particle stays in the vessel, the volume $V(t)$ of new liquid must enter, $V(t)$ being the total liquid volume at the time that the particle leaves the vessel. In other words, all 'old' material must have left the vessel when the observed particle is at the outlet of the vessel.

By differentiating equation (4.48) it is easy to derive

$$\dot{T}_d(t) = 1 - \frac{Q_o(t)}{Q_i(t - T_d(t))} \tag{4.49}$$

from which the delay function can be calculated numerically.

Proposition 4 (Section 4.1) can now be used to investigate, whether the system is z-invariant with respect to a modified time scale. But for the obvious choice

$$z = f(t) = \int_{t_0}^{t} \frac{Q_i(\tau)}{V(\tau)} d\tau \tag{4.50}$$

it follows that

$$f(t) - f(t - T_d(t)) = \int_{t-T_d(t)}^{t} \frac{Q_i(\tau)}{V(\tau)} d\tau \tag{4.51}$$

which is not constant in general, note (4.48). Equation (4.48) might suggest the new scale

$$z_1 = f_1(t) = \frac{1}{V(t)} \int_{t_0}^{t} Q_i(\tau) d\tau \tag{4.52}$$

but it is easy to check that this does not lead to a z-invariant representation either. Moreover, the scale (4.52) is not generally an increasing function, which means that the correspondence between the time scale and z-scale is not necessarily one-to-one.

But consider the scale $z_t = f_t(t)$, in which $f_t : \tau \mapsto f_t(\tau)$ is defined on the time interval $[t_0, t]$ such that

$$f_t(\tau) = \frac{1}{V(t)} \int_{t_0}^{\tau} Q_i(\nu) d\nu \tag{4.53}$$

The function is monotonously increasing (the flow rate and liquid volume are assumed to be positive as usual), and it is possible to change the independent variable in (4.46)

$$\begin{aligned}
\bar{y}(z_t) &= \bar{y}(f_t(t)) = \bar{u}(f_t(t - T_d(t))) = \bar{u}(\frac{1}{V(t)} \int_{t_0}^{t-T_d(t)} Q_i(\nu) d\nu) \\
&= \bar{u}(\frac{1}{V(t)} (\int_{t_0}^{t} Q_i(\nu) d\nu - \int_{t-T_d(t)}^{t} Q_i(\nu) d\nu)) \\
&= \bar{u}(z_t - 1)
\end{aligned} \tag{4.54}$$

By noticing that

$$z_t = \frac{1}{V(t)} \int_{t_0}^{t} Q_i(\nu) d\nu = z_1 \tag{4.55}$$

it follows that

$$\bar{y}(z_1) = \bar{u}(z_1 - 1) \tag{4.56}$$

which might suggest that the system is z-invariant in a similar way as described earlier.

However, it must be pointed out that the modified time scale has now been defined by means of the function $f_t(\cdot)$, which is defined on the time interval $[t_0, t]$. When moving from time instant t to $t + dt$ the values of the function according to (4.53) change on the whole domain $[t_0, t + dt]$. The true meaning of that can be explained as follows:

Consider equation (4.56) which is understood to mean that the system is z-invariant with respect to the scale z_1. The output concentration at time t can be calculated by looking at the input concentration at time $z_1 - 1$. The whole idea of z-invariant systems in general is in the simple input-output characteristics of a system, when modified time scales and the corresponding concentrations are used. The real time t is needed only to calculate the modified scale as time goes by; the input concentration values are tabulated as a function the new scale. The values of real time can be forgotten.

The situation changes, if relationship (4.56) is used. To calculate the output concentration at time t, the input concentration at the modified time $z_1 - 1$ has to be determined. But what is the input concentration at time $z' = z_1 - 1$? It is difficult to know, because at time t the whole past history of z has been re-scaled, and the old values of z have been forgotten. That is a direct consequence of the definition of the function (or actually its domain) and shows that the relationship (4.56) in this case must be seen as a formal result only. The plug flow vessel in the case of variable volume is not truly z-invariant in the true meaning of the concept.

The function $f_t(\cdot)$ is sometimes called *restriction* of $f_1(\cdot)$, because of the changing definition domain (Zenger, 1992). The origins of the concept comes from the function $f_1(\cdot)$, (4.52), which has been mentioned in the literature in the context of dealing with delay models or velocity profile models of processes with varying liquid volume.

Chapter 5

Tracer Tests

In process industry the determination of the RTD is a widely used and often the only possible practical modelling technique. Originally developed by Danckwets (1953) it still is a potential topic of interest especially among the practitioners and scientists in the field of chemical engineering (Thereska *et al.* 1996). In practice the RTD is determined experimentally by a tracer test, in which an amount of chemical or radioactive substance it injected at the input of the flow system impulsewise; the concentration of the tracer is then continuously measured at the outlet of the system. After a proper treatment of the output data the RTD is obtained.

In the previous chapters a systematic method has been developed to change the time-varying RTD of a flow system into a form suitable for analysis using the classical theory of time-invariant linear systems. The method can be interpreted as a complexity reduction in a more general setting, because the RTD yields the input/output characterization of the continuous flow process. In practical processes the measurements of the flow rate and liquid volume are usually available, implying that the use of the modified time scale in process analysis and controller design can easily be implemented in the process computer.

To test the method in practice a laboratory-scale pilot plant was used. Three different process vessels were constructed, and their RTDs under unsteady flow conditions were determined by using both chemical and radioactive tracers. The modified time scale was then introduced to test the validity of the invariance of the weighting function or RTD as predicted in the theory. These tests and their results are described in the current chapter.

[1]The content of this chapter is essentially the same as in (Niemi *et al.*, 1998).

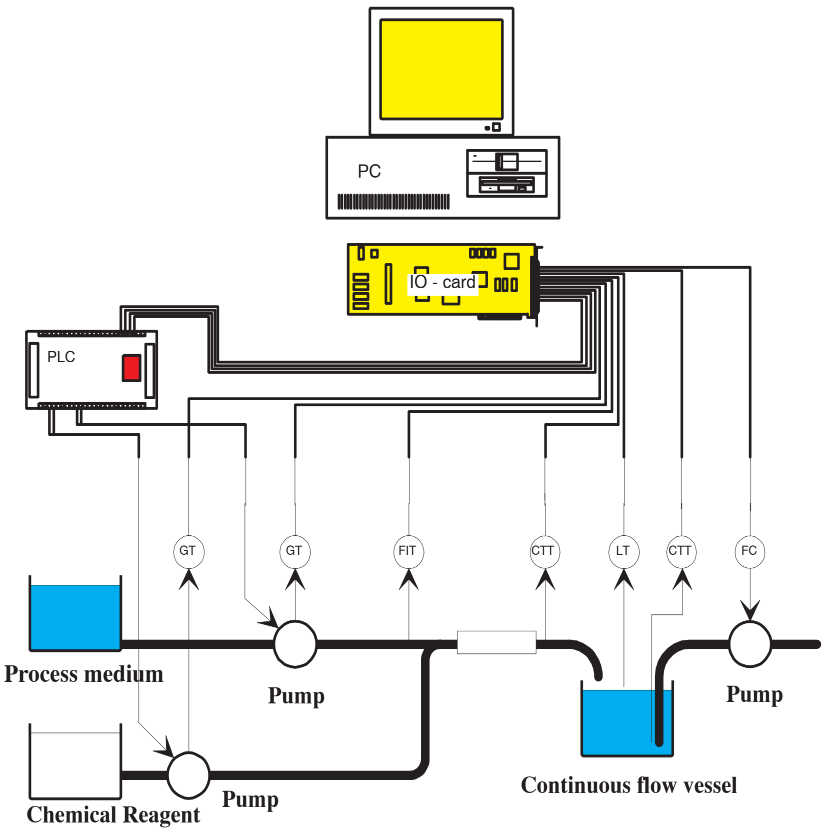

Figure 5.1: Pilot system for testing and control of continuous flow vessels under variable flow and volume

5.1 A laboratory-scale pilot system

A pilot system (Fig. 5.1) was used for practical testing of the results presented in the previous chapters. Three process vessels were built each having one feed channel and one outlet channel. Analyzers were used at the inlet and at the outlet of the vessel; the analyzers may be radiation probes, pH meters etc. A raw stream of tap water or other liquid was pumped from a storage vessel to the plant. A minor stream of a chemical agent was added to and mixed with it, in order to establish the final feed to the plant. The flow rate of the main stream was measured by an electromagnetic flow meter and that of the minor stream by a sensor for the stroke length of the chemical pump. A level height sensor delivered a signal which is proportional to the volume of liquid in the plant.

The measured signals and controls of the two pumps as well as the one at the outlet were stored in a PC computer. An interface card was used to transmit digital output data and analog input data between the computer and the pilot. Moreover, a programmable logic controller (PLC) was used to produce physical control signals to the actuators. The

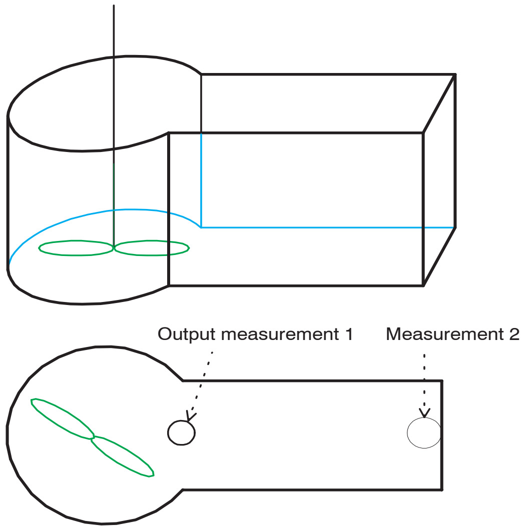

Figure 5.2: Flow vessel with mechanical agitator

necessary programs for conducting the tests were programmed for PC in C-language. As far as the radioactive tracer tests are concerned, an additional PC was used to collect data from the radiation gauges. All the time care was taken to make sure that the combination of data provided by the two PC's took place correctly without any timing errors. The first of the tested process vessels is shown in Fig. 5.2. It consists of a well-mixed part with a circular cross-section and a rectangular part. The output measurement can be done either at the well-mixed section or at the real outlet of the vessel.

A number of test series of different types was made for the determination of the RTD of the plant under constant and variable conditions. In each test, the liquid volume and the total feed flow were held constant or varied in a pre-programmed manner by the computer control system. In the first tests, single pulses of chemical tracer were injected under different, constant conditions, and the pH of the outlet liquid was measured and used for derivation of the RTD. Continuously variable chemical concentration was produced in the tests of the second series by means of the chemical agent stream, and the RTD was identified on the basis of pH measurements at the inlet and outlet, under constant and variable flow rates. Thirdly, radioisotope tracer pulses were injected under constant and variable flow rates and volumes, and the RTD was obtained by appropriate transformations of the outlet activities measured. Conductivity sensors were used in additional series of pulse tests which were made together with radiotracer tests or alone.

5.2 Tests with a chemical tracer

The plant was tested for its RTD with a short pulse of high concentration. The process liquid was tap water acidified with HCl, and each input pulse consisted of 4 ml of a strong NaOH solution injected in less than 3 seconds into the feed pipe. The titration curve of the process liquid was determined before the tests and its numerical equivalent was stored in the computer. This was used in the tests for conversion of the pH values measured at the outlet to Na^+ -ion concentrations which were only due to the added base. These were then converted to RTD readings by dividing them with the integral of the output. The tests were made under stationary operation conditions (constant flow rate and liquid volume), and the results were presented as functions of the dimensionless time variable $\theta = t/\bar{t}$, which under stationary conditions is equivalent to z, see (2.41).

In the first test the liquid volume was kept constant, but the flow rate had three constant values. Figure 5.3 shows the RTDs obtained (solid line: Q=550ml/min, dashed line: Q = 760 ml/min, dashed-dotted line: Q =900 ml/min, V=1200 ml in all cases). The pH sensor was positioned inside the vessel (at the output of the presumed well-mixed area), see Fig. 5.2.

The results in Fig. 5.3 show, similarly as the results obtained by Niemi *et al.* (1993), that the use of the dimensionless time variable brings the three RTDs close to each other. Although the deviations and their origination in errors of measurement or changes of flow pattern have not been evaluated as values of a quantitative criterion, the results suggest the same degree of invariability in the variable flow case as well, if the variations of flow keep within the same range. This means that argument θ of the RTD may be replaced with z.

Two additional test results (Figs. 5.4 and 5.5) show a similar degree of invariability when presented as a function of variable z. Note that Fig. 5.4 describes tests in which the output sensor was placed at the end of the cylindrical part of the vessel (well-mixed area). The vessel had a rectangular section following the cylindrical part, and in the later tests (Fig. 5.5) the output sensor was placed at the outlet of this section.

5.3 Identification by means of random signals

The process of flow and mixing in a continuous flow plant is normally linear with regard to concentrations of the material components involved. If the flow rate and volume are constant, the material transport dynamics can be represented by a constant parameter RTD or weighting function. The relationship from the concentration at the inlet to that

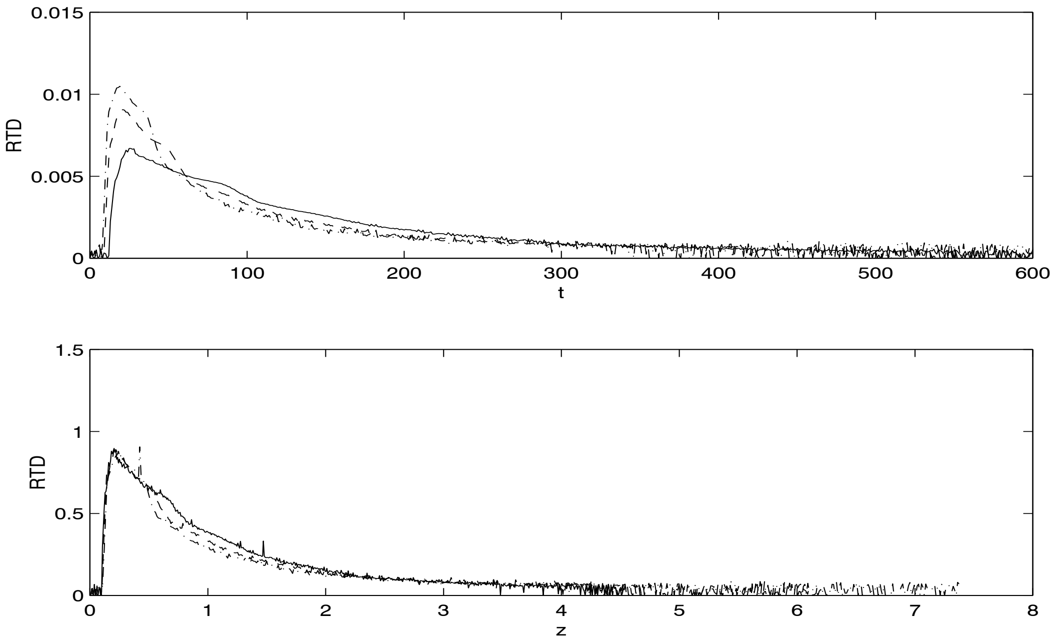

Figure 5.3: Chemical tracer tests (Q=550 ml/min (solid), 760 ml/min (dashed), 900 ml/min (dashed-dotted), V=1200 ml). In upper picture time scale in seconds; RTD scale in 1/s

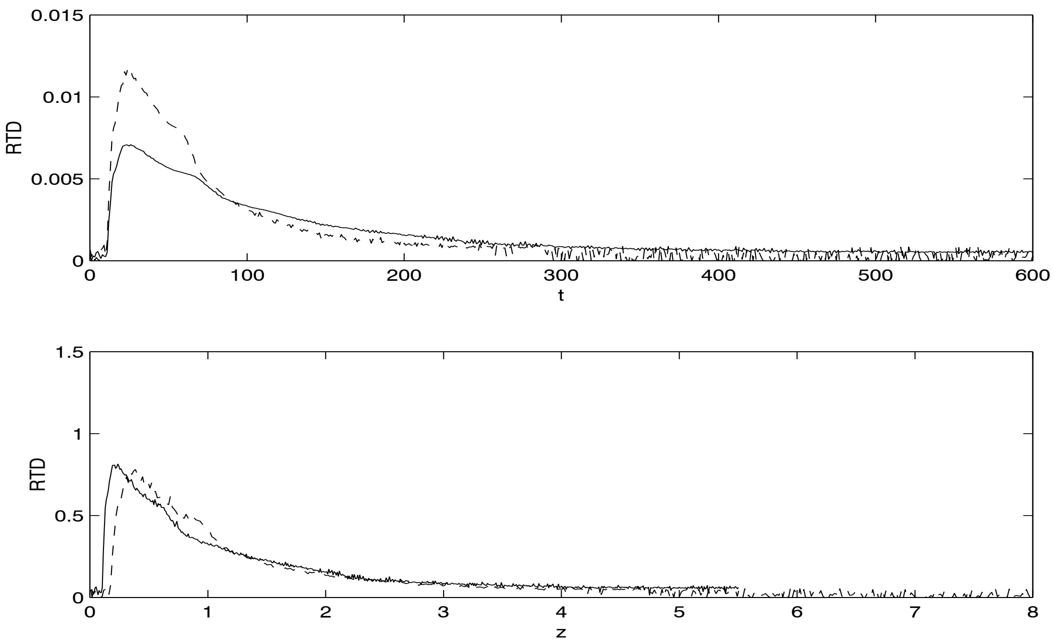

Figure 5.4: Chemical tracer tests (Q=550 ml/min, V=980 ml (solid), 550 ml (dashed)

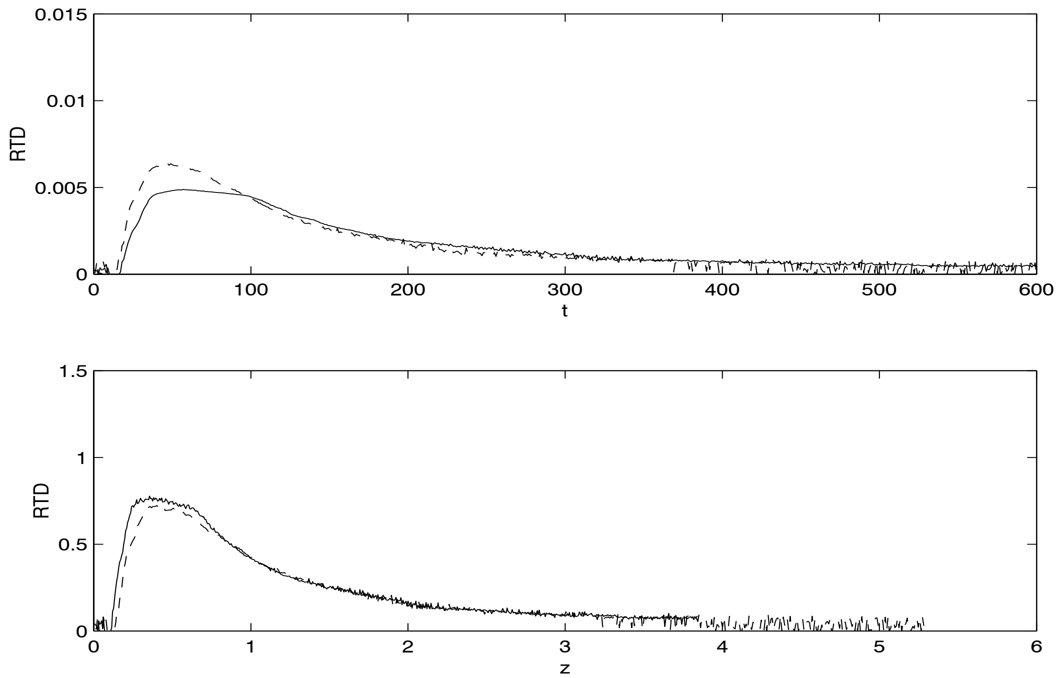

Figure 5.5: Chemical tracer tests. Measurement from the outlet of the extended vessel (Q=550 ml/min (solid), 760 ml/min (dashed), V=1400 ml)

at the outlet is expressed by convolution (5.1), which in terms of discrete time obtains the form (5.2). The function p' is here truncated to a finite impulse response (FIR), at a time $N\Delta t$ after which its value is negligible.

$$c(t) = \int_{-\infty}^{t} p'(t-\nu)c_i(\nu)d\nu = \int_{0}^{N\Delta t} p'(\nu)c_i(t-\nu)d\nu \tag{5.1}$$

$$c(t) = \sum_{i=0}^{N} p'(i\Delta t)c_i(t-i\Delta t)\Delta t \tag{5.2}$$

In the above equations the notation $p'(\nu)$ is used instead of $p'(\nu, 0)$. General methods have been developed which can be used for process identification by means of continuous, random concentration signals measured at the inlet and outlet. Such a method is the recursive parameter estimation algorithm by Niemi *et al.*(1993), which has been applied experimentally in the pilot plant process described above.

In these tests, the raw feed flow was held constant, while a continuous variable stream of chemical agent (NaOH) of about 100-fold concentration was added to and mixed with it. The chemical pump was controlled with a white noise signal generated by random noise software of the computer. The input and output concentrations were obtained by pH measurements and conversions. The estimation algorithm was repeatedly used for identification of the RTD of the plant as a sequence of 31 parameters, i.e. values of

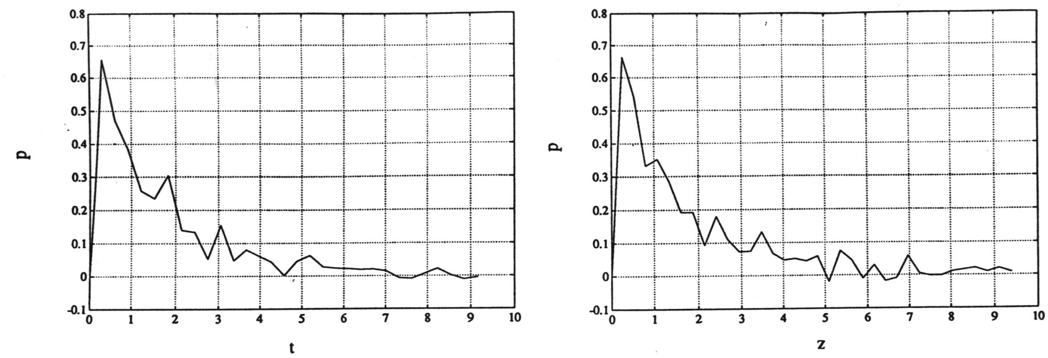

Figure 5.6: Identified responses under constant and variable flow (Niemi *et al.*, 1993)

$p(i\Delta t)$, $(0 \leq i \leq N; N = 30)$. After an initial period of less than 20 min, the estimates converged, and the data in Fig. 5.6 represent the averages of their 150 last values, with θ as argument. In the figure one time unit corresponds to 50s.

Under variable flow, the weighting function model of the plant is described by a linear, variable parameter model $p'(t, \nu)$, and the functional relationship of its input and output concentrations has the form of (5.3). The resulting input/output relationship in discretized form is given in (5.4).

$$c(t) = \int_{-\infty}^{t} p'(t, \nu) c_i(\nu) d\nu \tag{5.3}$$

$$\bar{c}(z) = \sum_{i=0}^{N} \bar{p'}(i\Delta z) \bar{c_i}(z - i\Delta z) \Delta z \tag{5.4}$$

Note that the latter equation holds under the assumption that the system is z-invariant, see equations (2.29) and (2.32) in Chapter 2. The parameters of $\bar{p'}(i\Delta z)$ can be estimated in the same manner as those of $p'(i\Delta t)$, for equi-spaced values of z; for details, see (Tian, 1994). In order to accomplish the identification in practice, the flow rate in the system of Fig. 5.1 was measured in real time and integrated for production of the needed values of the modified time scale argument. The variable flow rate of the process medium was produced under pre-programmed control of the main pump. The variation was typically sinusoidal with the mean of 750 ml/min and amplitude of 225 ml/min. The chemical pump was controlled by a similar, parallel signal needed for production of a constant mean value of the final input, plus a white noise signal of zero mean superimposed with the former one. The resulting input and output concentrations were obtained by pH measurements and conversions. The parameter estimation procedure converged similarly as in the previous test described producing the RTD of Fig. 5.6 as a function of z.

The general form of the function is very similar to that obtained under constant flow. The identifications thus show that the flow pattern of the plant is not observably affected by the variation of flow rate within a wide range of amplitude and that the integrated flow variable z takes correctly into account the variation.

The RTDs produced by recursive parameter estimation show random fluctuation which is typical to results produced by such discrete estimation methods, but agree well, for most part of their course, with the responses of the same process vessel obtained with chemical tracer (Figs. 5.3-5.5). The systematic difference of the identified RTDs and the less peaky pulse test results at low values of the argument seem to be connected with large changes of the concentration in the pulse tests which the pH sensor is not able to follow due to its slow dynamics. Another possibility is the diffusion of the highly concentrated chemical constituting the pulse. The flow variation is not a probable cause of the difference, because the models identified under constant and variable flow are practically coincident here.

5.4 Tests with radioisotope tracers

The same plant was submitted to tests with small amounts of radioisotope tracer. Tc-99 was obtained from a Mo-Tc isotope generator in sodium pertechnetate solution and was considered a suitable tracer element due to its short half-life and low gamma energy, and because of its known uses in medical diagnosis. A dose of 18MBq in 0.5 ml of liquid which could be injected within less than a second was found sufficient after some preliminary testing. The two scintillation detectors were provided with collimators and located at the same points as the pH detectors in the earlier tests; the first one was used only for the zero-time signal. The flow rate and level height meters were calibrated for computation of the modified time scale.

Steady state tests were first made at a fixed volume for different, constant values of feed flow rate. Their results were shown as functions of θ or z, examples of which are presented in Fig. 5.7 (Q=550 ml/min (solid), Q=760 ml/min (dashed), Q=900 ml/min (dashed-dotted), $V = 1200$ ml in all cases). The results of a test in which both the flow rate and liquid volume varied sinusoidally are presented in Fig. 5.8 ($Q_i = 650 + 200 sin(2\pi t/T)$ ml/min, $V = 980 + 150 sin(2\pi t/T)$ ml , T=50 min, (solid line), T=25 min (dotted line), T=15min (dashed-dotted line)). The measurement was made at a point inside the vessel (at the output of the presumed well-mixed area). The responses were very similar in all the cases tested. A perfect mixer model including a delay factor as adjustable parameter was fitted to the RTDs obtained. The value of the delay was chosen for the best fit over the range of variation of z in the tests. These and later tracer tests with the same plant have proved that this operates in the manner of a perfect mixer preceded by a small plug flow element of z_d of about 0.06, corresponding to the pipeline before the mixer. A second test series was carried out by taking the measurement from the final output of the vessel. The RTDs are presented in Fig. 5.9 and the flow rates and volumes in Fig. 5.10. The solid and

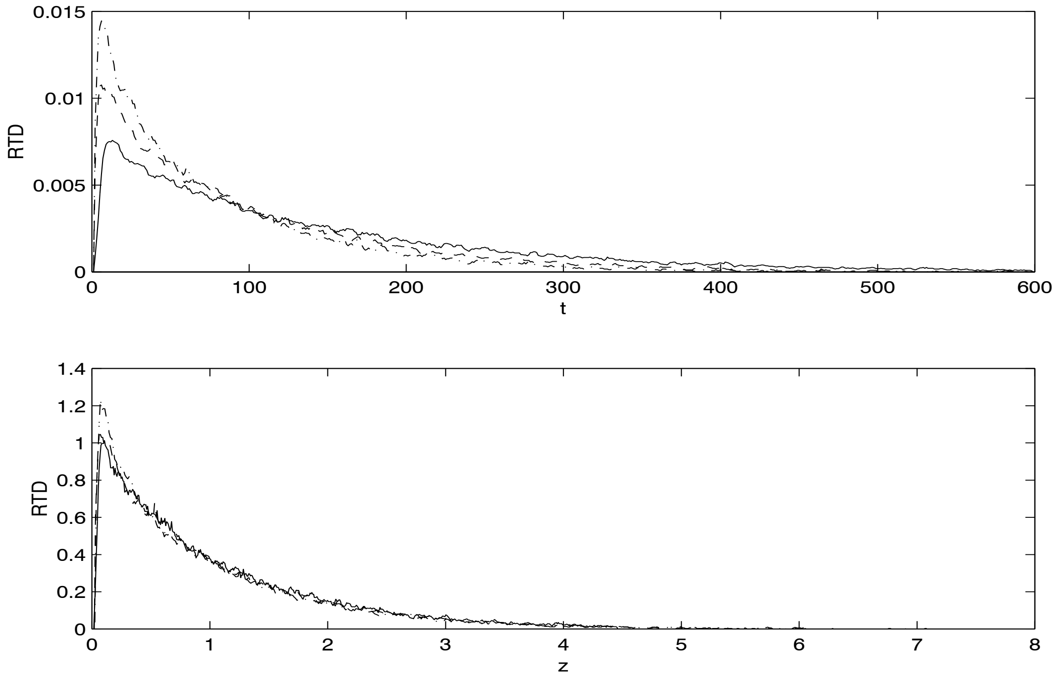

Figure 5.7: Radioisotope tracer tests under steady state conditions

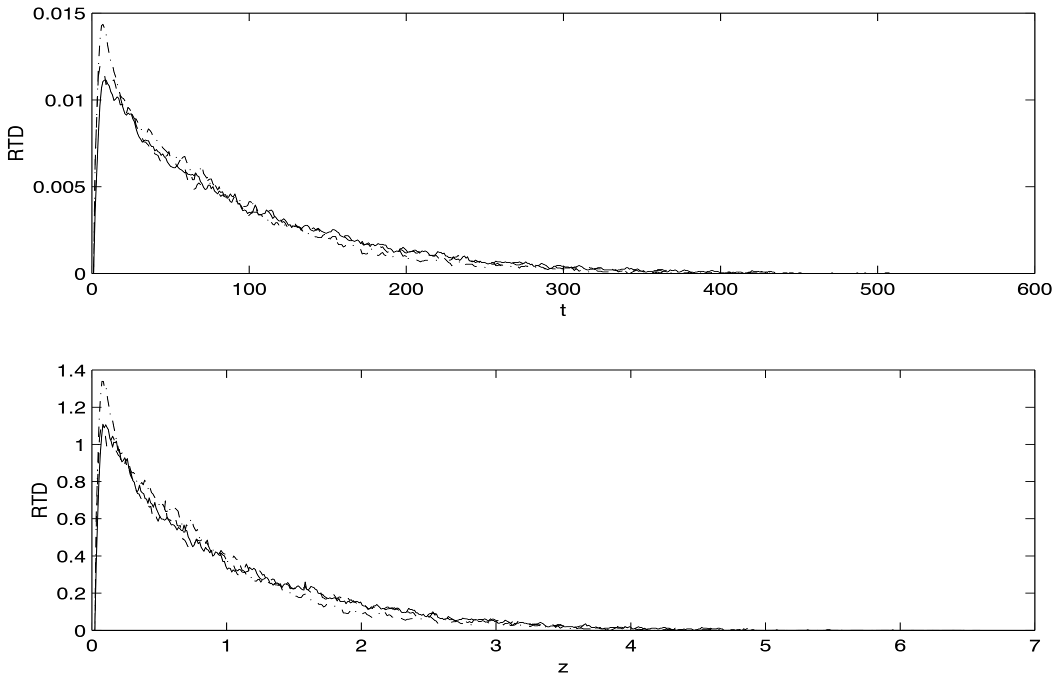

Figure 5.8: Radioisotope tracer tests under sinusoidal flow rate and volume variations

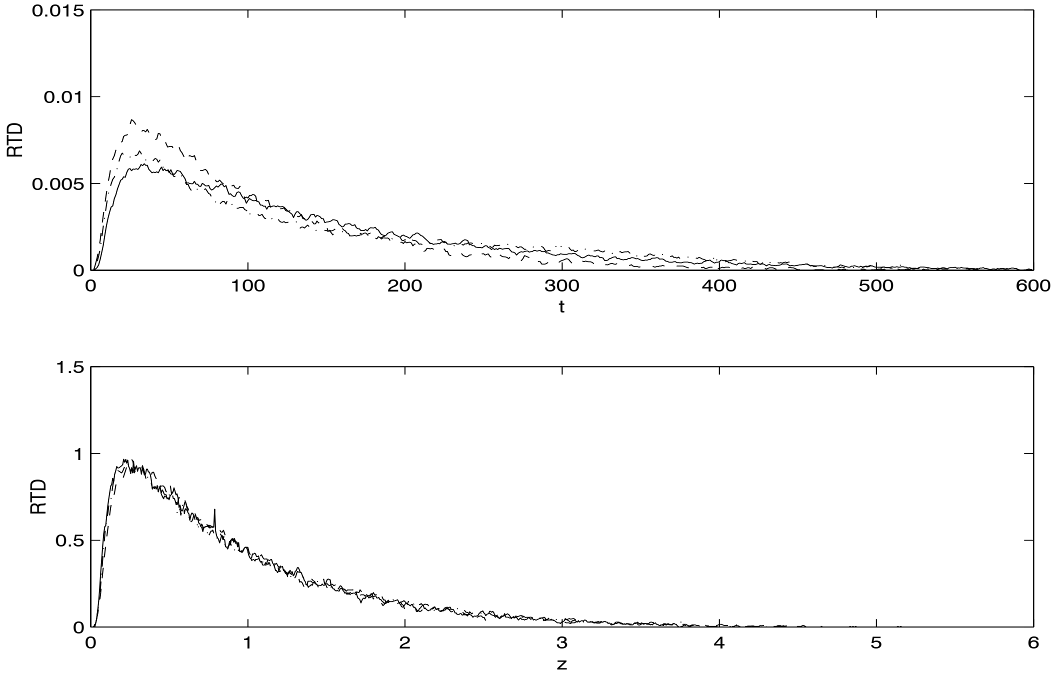

Figure 5.9: Radioisotope tracer tests. Measurement at the output of the extended vessel

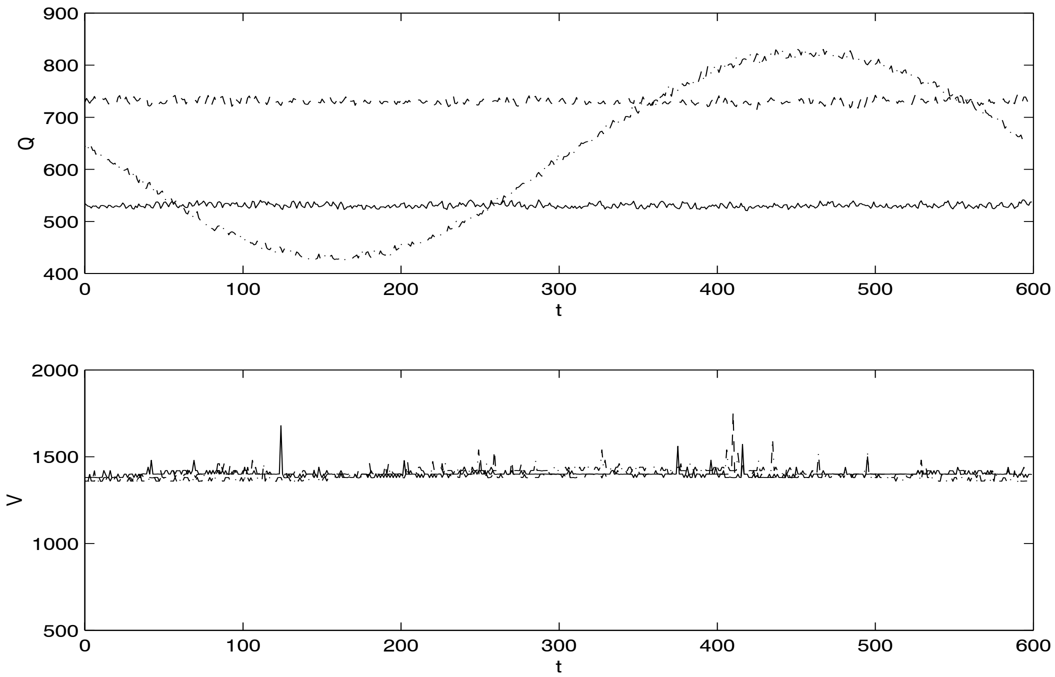

Figure 5.10: Radioisotope tracer tests. Flow rates and liquid volumes

dashed lines represent tests with different constant flow rates and constant liquid volume (Q=550 ml/min and 760 ml/min, V=1400ml). The dashed-dotted line represents a test with sinusoidally varying flow rate and constant liquid volume ($Q_i = 650 + 200 sin(2\pi t/T)$ ml/min, T=10 min, V = 1400ml). The uniformity of the results shows the usefulness of the variable z under variable flow conditions, and in the case of variable volume and feed flow in the almost perfectly mixed vessel, at least if the parameters change slowly. It is also evident that the flow patterns of the plant are not observably affected by variation of flow rate and volume within a wide range.

The results are generally similar but clearly more accurate than those obtained by means of continuously variable, random concentration signals. They are now recorded at much shorter intervals and the random scatter is less. The peak of RTD is higher and steeper and has obviously been reproduced better by the radioisotope tracer. This is related to the small sample volume within the liquid and to low background achieved with a properly collimated and shielded detector, which contributed to the accuracy of the radiotracer method.

The results differ even more from those obtained with the chemical tracer method. The latter proved unable to follow fast concentration changes delivering, at least if used as described previously, a very coarse picture of the real RTD, even if this is independent of the flow rate. It is obvious that the dynamics of the count rate meter is much faster than that of the pH meter, and that, because of the low chemical concentration of the radioisotope tracer pulse, diffusion has a negligible effect on the response which it generates.

5.5 Tests with radioisotope and chemical tracer

An oblong open vessel was constructed aiming at an approximately laminar velocity profile of the liquid and another series of tests was carried out (Fig. 5.11). The pH transducers were substituted with conductivity sensors which were then used, together with nuclear gauges, for recording of the responses to simultaneous injections of the salt solution and the radiotracer. Fig. 5.12 shows the results for the stationary states of Q=500 ml/min (solid line), Q=700 ml/min (dashed line), Q=800 ml/min (dashed-dotted line), $V \approx$2100 ml in all cases. The general form of the responses resembles to some degree that of the theoretical laminar flow vessel (Niemi, 1990), but is lower and less steep at both sides and shows less of time delay. A dependence of the input/output flow pattern on the stationary point of operation is in this case shown by the differing course of one of the three RTD functions of z (case of low flow rate). Deviation of the internal flow pattern from laminar flow was then demonstrated by means of a dye. It thus turned out, after some improvements of the vessel had been tried, that the establishment of a laminar flow pattern in the major part of an open, pilot size vessel in laboratory meets difficulties. The results

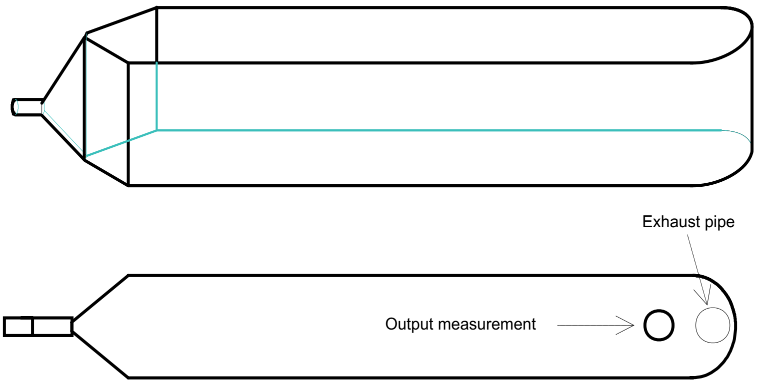

Figure 5.11: A continuous flow vessel with no mixer

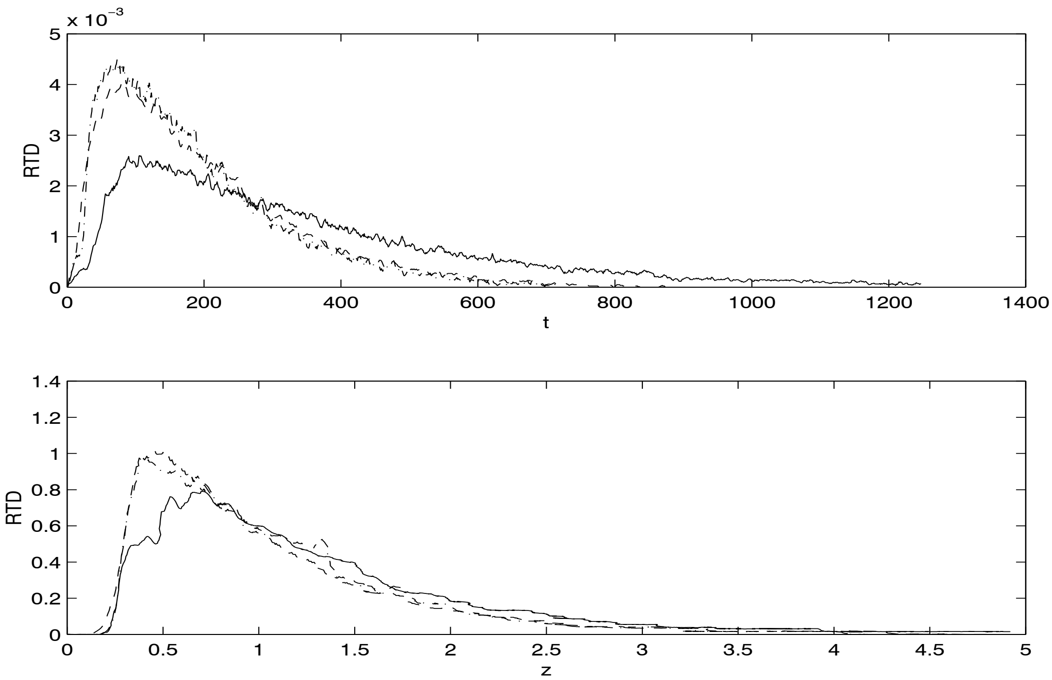

Figure 5.12: Test responses of an oblong open vessel

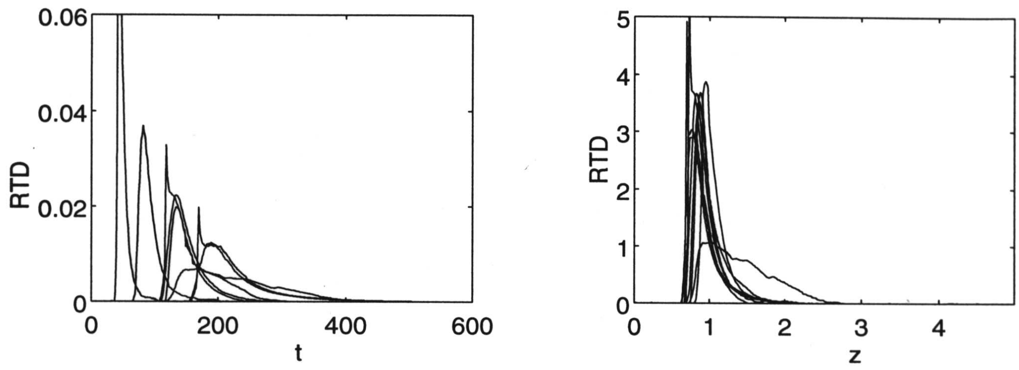

Figure 5.13: Tests with a tubular vessel (Zenger, 1995)

of this test series showed however that appropriate tracer conductivity measurements were more accurate than the use of pH sensors, the somewhat slow dynamics of which can cause distortion of the shape of response. The use of salt solution as tracer and conductivity measurements could, at least in this case of a single electrolyte in water, well be used as a feasible alternative to radiotracer test.

5.6 Tests with a tubular vessel

The results of the previous tests were the reason for trying to set up a process, in which a laminar flow pattern could be accomplished more accurately. A 10 meter long rubber tube 1 cm in diameter was chosen as process vessel. Low flow rates of 200 to 500 ml/min were used; turbulent conditions could be expected in only one test. The measurements were done by using water from an ion-exchange as process flow, into which 1 ml of salt solution was injected at the inlet of a premixer preceding the vessel. The tracer concentration was varied during the test series, because it was not known what tracer concentration would give the best accuracy of the output measurement. In Fig. 5.13 all RTDs obtained are shown as functions of time and volumetric scales. It is noticed that the curves become reasonably close to each other if expressed in z-domain. Their shape is now closer to the theoretical response: the height of many peaks is near to $p = 4$ and the delay near to $z = 0.5$ being always somewhat larger, which refers to a real delay in series with the laminar flow component. The only exception appearing clearly in the figure was the result of the test with highest amount of the tracer which obviously did not mix properly in the premixer.

5.7 Conclusions of the tests with the pilot plant

The RTDs of a continuous flow process can be expressed, under changing flow rate and liquid volume, as unambiguous function of an integrated variable introduced previously, if the flow pattern is not affected by the changes. Such a general RTD can be derived from the regular RTD for stationary conditions, but it can also be determined experimentally, if the flow and volume are measured on line during the test.

The pilot system described enabled to vary flow rates and liquid volume under computer control in chosen, arbitrary manners, and likewise the input concentration, in addition to a pulse injection of the tracer. All relevant process variables were measured and data transferred to a computer in real time, in order to obtain the RTD as a function of the appropriate integrated variable. It turned out that such RTDs of the well mixed vessel tested were, under constant and varying flow and volume, almost identical and in agreement with the perfect mixer model accompanied by a minute delay and expressed in terms of the same variable. In the extended vessel tested, the RTDs measured differed from that of a single perfect mixer, but the mixing was still good and the change of variable brought the responses close to each other, despite the changing flow rate. Similar conclusion applies to a vessel whose input/output flow pattern is close to that of the laminar flow vessel.

The radioisotope tracer and electrolyte tracer measured by conductivity were well suited to laboratory testing for RTDs. The measurement of a chemical tracer by pH is applicable to on-line identification, but is inaccurate in pulse testing. The use of chemical analyzers of other types and corresponding tracers was not studied. The presented methods have potential to full scale application, since although the flow rate and volume or level height of liquid are variable in industrial plants, they are usually measured and therefore available for extraction and use of RTDs in real time. Radiotracers are obviously least sensitive to disturbances but practically limited to pulse testing of such processes. The use of chemicals and measurement of pH or conductivity for identification is often effectively limited by various dissolved and particulate components of the process medium.

Chapter 6

Controller Design

The ideas developed in the previous chapters can be applied in the controller design of such flow processes, in which the flow rate and liquid volume are variable. If the parameters of the model are time-varying, it is difficult to design a controller that performs well in all operation conditions. In traditional control design various methods have been used to solve that problem. Specifically, the principle of *gain scheduling* has been widely used, see e.g. (Åström and Wittenmark, 1995), (Niemi *et al.*, 1990). The idea in gain scheduling is that the controller is tuned at several operation points to form a schedule, which is then used to change the controller parameters continuously as the operation point of the process changes. The use of the method requires that the operation point can be measured.

Controller design based on the use of the modified time scale has a close relationship to the concept of gain scheduling. If a flow process model is z-invariant, the controller can be designed to operate in z-domain to eliminate the effects of varying flow rate and varying volume. The values of these quantities must be measured and/or calculated in order to determine the z-variable. The control law is then transformed back into time domain, which leads to an algorithm in which the parameters are continuously changing, when the flow rate and volume measured from the process are changing. The change of the parameters takes place automatically, and no pre-programmed schedule is needed. Only the basic tuning of the controller in a nominal operation condition is needed.

In the current chapter a PID controller with time variable parameters is developed. Examples on how to construct suitable discrete-time versions of the algorithm for a possible use in a digital automation system are given. An example of a practical application is presented, and the performance of the algorithm is demonstrated.

The results concerning the PID controller and its modifications are extremely important from the practical viewpoint. For example, according to Åström and Hägglund (1995) more than 95 % of unit controllers in process industry are of PID or PI type. Many of these have been tuned badly, so that the operation of the closed loop system is far from being optimal. In an example case reported by Bialkowski (1993), a process with 2000 control loops was investigated. 97 % of these were of PI type, and 20 % of them were found to work well. Reasons for bad operation were mainly because of poor tuning and actuator problems. Other problems reported have been e.g. sensor malfunctions, bad choice of the sampling rate or antialiasing filters. There are even studies claiming that a large percentage of controllers are operated in manual mode or use the factory tuning, meaning that they have never been tuned for the process they are used to control!

The necessity to provide automatic tuning methods is obvious. The time-varying controller to be presented below does not provide the basic tuning, but it helps to keep the tuning correct in the case of time-variable disturbances affecting the process.

Another application from the field of optimal control is discussed in the last part of the chapter.

6.1 A time-variable PID controller

Consider a mixing process with variable flow and volume. If the system model is z-invariant, the time-varying characteristic of the model can be removed by writing the model equations in z-domain. The controller can then be designed and tuned in z-domain to give a good performance of the closed-loop system. The controller is realized by transforming the control algorithm back to time domain.

The method can be motivated by using the following heuristic reasoning: For every interval T_z ($z \in [z_0, z_1]$) there is a unique interval T_t ($t \in [t_0, t_1]$) such that the model variables of the system obtain exactly the same values within T_z and T_t, respectively. If the closed loop performance of the system is good in T_z, it can be expected to be good in T_t as well. Hence, the design of a controller in z-domain for a z-invariant system seems to be a reasonable procedure, because the time-varying nature of the process is implicitly included in the z-variable and standard design methods can easily be applied in z-domain.

However, it should be pointed out that the presented strategy is based on heuristic reasoning, and it therefore deserves criticism. Although the performance of the closed loop system would be optimal in z-domain with respect to some criterion, the optimality in time domain according to a similar criterion is not necessarily achieved. For large variations of the flow rate the z-variable may deviate so much from the time variable that a good performance in z-domain does not mean a desired performance in time domain.

However, if the flow rate varies within reasonable limits only, the design procedure seems to be applicable.

Let us now use the design method to construct a time-varying PID controller. If the process model is z-invariant, the controller can be given in that domain as

$$\bar{u}(z) = K_{pz}\bar{e}(z) + K_{iz}\int_0^z \bar{e}(\xi)d\xi + K_{dz}\frac{d\bar{e}(z)}{dz} \tag{6.1}$$

where $\bar{u}$ is the controller output, $\bar{e}$ is the controller input (error signal) and K_{pz}, K_{iz}, K_{dz} are the (constant) gains of the controller. By using the familiar transformations $z = f(t)$, $t = h(z)$ again, the control algorithm can be written in time domain. For the three different terms the equations are

$$K_{pz}\bar{e}(z) = K_{pz}e(h(z)) = K_{pz}e(t)$$

$$K_{iz}\int_0^z \bar{e}(\xi)d\xi = K_{iz}\int_0^z e(h(\xi))d\xi = K_{iz}\int_{t_0}^t e(\tau)\dot{f}(\tau)d\tau$$

$$K_{dz}\frac{d\bar{e}(z)}{dz} = K_{dz}\frac{de}{dz}(h(z)) = K_{dz}\frac{de}{dh}(h(z))\frac{dh}{dz}(z) = K_{dz}\frac{de}{dt}(t)(\frac{df}{dt}(t))^{-1}$$

Using the z-variable

$$z = f(t) = \int_{t_0}^t \frac{Q(\nu)}{V(\nu)}d\nu$$

the control algorithm becomes in time domain

$$u(t) = K_{pz}e(t) + K_{iz}\int_{t_0}^t \frac{Q(\nu)}{V(\nu)}e(\nu)d\nu + K_{dz}\frac{V(t)}{Q(t)}\frac{de(t)}{dt} \tag{6.2}$$

The same controller expression was originally derived by Niemi (1991).

The algorithm is time-varying, because the flow rate $Q(t)$ and the liquid volume $V(t)$ have an effect on the gain terms. In fact, the controller can be implemented by writing the standard PID controller algorithm as

$$u(t) = K(t)[e(t) + \int_{t_0}^t \frac{e(\nu)}{T_i(\nu)}d\nu + T_d(t)\frac{de(t)}{dt}] \tag{6.3}$$

where the coefficients are continuously modified as

$$K(t) = K_{pz} \tag{6.4}$$

$$T_i(t) = \frac{K_{pz}V(t)}{K_{iz}Q(t)} \tag{6.5}$$

$$T_d(t) = \frac{K_{dz}V(t)}{K_{pz}Q(t)} \tag{6.6}$$

Note that if a PID algorithm is realized in this way, the integral action is implemented according to equation (6.3), so that the coefficient $T_i(t)$ is part of the integrand.

It is interesting to note that the algorithm operates like an ‘automatic’ gain scheduling controller, because the tuning parameters are continuously changed according to the flow measurement.

In the case that a discrete-time controller is desired, it is straightforward to discretize equation (6.2). If the integral part is approximated by a forward approximation and the derivative part by taking the backward difference, the result becomes

$$u(iT) = P(iT) + I(iT) + D(iT) \tag{6.7}$$

where

$$P(iT) = K_{pz} e(iT) \tag{6.8}$$

$$I(iT+T) = I(iT) + K_{iz}\frac{Q(iT)}{V(iT)}e(iT)T \tag{6.9}$$

$$D(iT) = K_{dz}\frac{V(iT)}{Q(iT)}[\frac{e(iT)-e(iT-T)}{T}] \tag{6.10}$$

In the above equations the absolute time instant is iT, in which T is the sampling interval and i is an integer value.

It is easy to derive similar results for many existing modifications of the PID algorithm. For example, let the controller algorithm be given in Laplace domain as

$$U(s) = K[E(s) + \frac{1}{T_i}\frac{E(s)}{s} - \frac{sT_d}{1+sT_d/N}Y(s)] \tag{6.11}$$

where E is the error signal, Y is the measured process output signal and U is the controller output. The tuning parameters are gain K, integration time T_i, derivation time T_d and the value N, which is used in the lag term of the derivative part. (According to Åström and Wittenmark (1997) this term is typically in the range 3-10.) Note that the inverse Laplace transformation of the controller algorithm now operates as a function of ‘time’ z.

Simple calculations show that the discrete time-varying algorithm is again of the form (6.7), where the P and I parts are

$$P(iT) = Ke(iT) \tag{6.12}$$

$$I(iT+T) = I(iT) + \frac{K}{T_i}\frac{Q(iT)}{V(iT)}e(iT)T \tag{6.13}$$

The derivative part fulfils the differential equation

$$\frac{T_d}{N}\frac{d\bar{u_d}(z)}{dz} + \bar{u_d}(z) = -KT_d\frac{d\bar{y}(z)}{dz} \tag{6.14}$$

which in time domain is

$$\frac{T_d}{N}\frac{du_d(t)}{dt} \cdot \frac{V(t)}{Q(t)} + u_d(t) = -KT_d\frac{dy(t)}{dt} \cdot \frac{V(t)}{Q(t)} \tag{6.15}$$

Discretization gives then

$$\begin{aligned} u_d(iT) &= \frac{T_dV(iT)}{TNQ(iT)+T_dV(iT)}u_d(iT-T) - \frac{KNT_dV(iT)}{TNQ(iT)+T_dV(iT)}y(iT) \\ &+ \frac{KNT_dV(iT)}{TNQ(iT)+T_dV(iT)}y(iT-T) \end{aligned} \tag{6.16}$$

and $D(iT) = u_d(iT)$ in (6.7).

The method of designing a z-invariant controller for a z-invariant process is easy to understand, because the resulting closed-loop system performs as a function of z like any time-invariant system. Transforming the controller algorithm back into time domain and discretizing it if desired are then straightforward operations.

A different approach has been presented by Andersson and Pucar (1995) and used e.g. by Åström and Wittenmark (1995). A time-varying system model has been changed into a discrete form with constant coefficients by using a time-variable sampling interval. A discrete-time controller with constant coefficients, which uses the same sampling intervals, can then be designed. To explain the method consider the system (3.1), where $A(t) = k(t)\bar{A}$, $B(t) = k(t)\bar{B}$. Assume that the functions $k(\cdot)$ and $u(\cdot)$ are constant during the intervals $iT \leq t \leq iT + T_t$, where the real number T_t denotes the (time varying) sampling interval. At time t the solution of the state equation can be written as

$$\begin{aligned} x(t) &= e^{\bar{A}k(iT)(t-iT)}x(iT) + \int_{iT}^{t} e^{\bar{A}k(iT)(t-\tau)}k(iT)\bar{B}u(iT)d\tau \\ &= e^{\bar{A}k(iT)(t-iT)}x(iT) + k(iT)\left[\int_{iT}^{t} e^{\bar{A}k(iT)(t-\tau)}d\tau\right]\bar{B}u(iT) \\ &= e^{\bar{A}k(iT)(t-iT)}x(iT) + k(iT)\left[\int_{iT}^{t}\sum_{l=0}^{\infty}\frac{1}{l!}\left(\bar{A}k(iT)(t-\tau)\right)^l d\tau\right]\bar{B}u(iT) \\ &= e^{\bar{A}k(iT)(t-iT)}x(iT) + k(iT)\sum_{l=0}^{\infty}\frac{1}{l!}\bar{A}^lk(iT)^l\left[\int_{iT}^{t}(t-\tau)^l d\tau\right]\bar{B}u(iT) \\ &= e^{\bar{A}k(iT)(t-iT)}x(iT) + \left[\sum_{l=0}^{\infty}\frac{\bar{A}^lk(iT)^{l+1}}{(l+1)!}(t-iT)^{l+1}\right]\bar{B}u(iT) \end{aligned} \tag{6.17}$$

At the time $t = iT + T_t$ the sampled system is

$$x(iT+T_t) = e^{\bar{A}k(iT)T_t}x(iT) + \left[\sum_{l=0}^{\infty}\frac{\bar{A}^lk(iT)^{l+1}}{(l+1)!}T_t^{l+1}\right]\bar{B}u(iT) \tag{6.18}$$

By using the sampling interval

$$T_t = \frac{\alpha}{k(t)} \tag{6.19}$$

where α is any nonzero real number, the resulting system becomes

$$x(iT + T_t) = e^{\bar{A}\alpha} x(iT) + \left[\sum_{l=0}^{\infty} \frac{\bar{A}^l \alpha^{l+1}}{(l+1)!} \right] \bar{B} u(iT) \tag{6.20}$$

which is independent of the function $k(\cdot)$. Note that the function $k(\cdot)$ has been assumed to be constant in the sampling interval. The interval T_t is determined from (6.19) in the beginning of the sampling period after which it is held constant until the next sampling instant.

The method can also be discussed by using the technique of the modified time scale. Consider equations (3.1), (3.2), (3.3), where $A(t) = k(t)\bar{A}$, $B(t) = k(t)\bar{B}$, $C(t) = \bar{C}$, $D(t) = \bar{D}$. Because the state, input, and output variables in (3.1) and (3.3) are in one-to-one correspondence, it is evident that the discretized forms of the realizations have a similar correspondence, too. Assuming a zero-order hold and a constant sampling interval in (3.3) leads to a discrete representation with constant coefficients. If a corresponding discretization is carried out in (3.1), the two systems are again in one-to-one correspondence with each other. Note however that a constant interval $T_z = \Delta z$ is time-varying (T_t) in time domain. Also note that the method is equally valid for all z-invariant systems, including those that contain delay.

There still remains the question of determining the correct sampling instant. In the above analysis based on the results of Andersson and Pucar (1995) the sampling interval was chosen to be inversely proportional to the function $k(\cdot)$. However, it was assumed that both the functions $u(\cdot)$ and $k(\cdot)$ are constant between the sampling instants. It is reasonable to look at that a little closer. Intuitively, the principle of 'inverse sampling' can be derived also by the following simple way. By the definition of the modified time scale it holds

$$\dot{f}(t) = d_1 k(t) \tag{6.21}$$

Approximating the derivative by a forward difference gives

$$f(t + \Delta t) - f(t) \approx d_1 k(t) \Delta t \tag{6.22}$$

where Δt is the sampling interval. Choosing Δt inversely proportional to $k(t)$ is now obvious to make T_z (approximately) constant. In practice, it is possible to synchronize the sampling mechanism with a positive displacement meter instead of a clock (Niemi, 1991).

Let the constant sampling interval in z-domain be

$$T_z = \Delta z = f(t + \Delta t) - f(t) = d_1 \int_t^{t+\Delta t} k(\nu) d\nu \tag{6.23}$$

The derivative must be zero

$$\frac{d(\Delta z)}{dt} = d_1 [k(t + \Delta t)(1 + \frac{d(\Delta t)}{dt}) - k(t)] = 0 \tag{6.24}$$

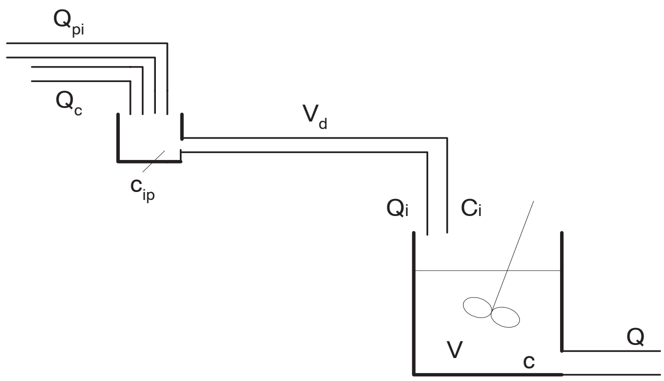

Figure 6.1: A simple concentration control system

and so

$$\frac{d(\Delta t)}{dt} = \frac{k(t)}{k(t+\Delta t)} - 1 \approx \frac{k(t)}{k(t) + \frac{dk(t)}{dt}\Delta t} - 1 \tag{6.25}$$

where the term $k(t+\Delta t)$ has been approximated with the first order Euler approximation $k(t+\Delta t) \approx k(t) + \frac{dk(t)}{dt}\Delta t$. But for the choice $\Delta t = \frac{\alpha}{k(t)}$ it follows that

$$\begin{aligned} \frac{d(\Delta t)}{dt} &\approx \frac{k(t)}{k(t)+\frac{dk(t)}{dt}\cdot\frac{\alpha}{k(t)}} - 1 = -\frac{\frac{dk(t)}{dt}\cdot\frac{\alpha}{k(t)}}{k(t)+\frac{dk(t)}{dt}\cdot\frac{\alpha}{k(t)}} \\ &\approx -\frac{\frac{dk(t)}{dt}\cdot\frac{\alpha}{k(t)}}{k(t)} = -\frac{\alpha\frac{dk(t)}{dt}}{k^2(t)} \end{aligned} \tag{6.26}$$

in which it has been assumed that the term $\frac{dk(t)}{dt}\frac{1}{k(t)}$ is small compared with $k(t)$. The result is noticed to be the same as the direct derivative of $\Delta t = \alpha/k(t)$.

However, the inverse sampling method is an approximation, which is accurate only if the function $k(\cdot)$ does not change much during each sampling interval. If, on the other hand, a constant sampling rate in time domain is used, the resulting equations are time-varying but exact. That is the case e.g. when the PID controller (6.2), (6.7) is used.

To demonstrate the performance of the time-varying PID controller consider the example presented by Åström and Wittenmark (1995), see Fig. 6.1. The system consists of a perfect mixer, where the liquid volume is constant although the flow rate can vary. The input to the system is the concentration of a chemical. However, there is a delay in the control (input) signal. The system can be described by

$$V\frac{dc(t)}{dt} = Q(t)(c_i(t-T_d(t)) - c(t)) \tag{6.27}$$

where c is the concentration in the mixer, c_i is the input concentration, $Q(t)$ is the flow rate, V is the volume of the mixer, V_d is the volume of the pipe that causes the delay in the concentration input, $T_d(t)$ is the delay function. The following simulations are done with different constant flow rates; in each case the delay function is then given by

$$T_d(t) = V_d/Q \tag{6.28}$$

The input concentration to the mixer can be modelled by combining the process flow and the flow of a reagent in a small vessel with negligible dynamics. The reagent is assumed to be so strong that the control flow is small compared with the process flow.

Notice that both the delay in the concentration input and the time constant of the process are inversely proportional to the flow rate. When the flow rate is small the control of the system becomes more difficult.

The simulations demonstrate the operation of the closed loop system, when a standard PID controller, equation (6.3) with $K(t) = K$, $T_i(t) = T_i$, $T_d(t) = T_d$ constants, or the time-varying PID$_z$ controller, equation (6.2), are used to control the reagent flow in order to obtain the desired concentration at the outlet of the mixer. The parameters of the process have been chosen such that nominally $Q = 1$ so that the time constant of the process is 1. The parameter V_d has the value 1. The simulations describe the closed loop response of the system, when a step change from 0 to 1 has occurred in the reference value of the concentration at the time 0.

Figure 6.2 shows four closed loop responses of the system. In each case the flow rate has been a different constant, and a PID controller with constant coefficients has been used. The parameter values of the controller are $K = 0.5$, $T_i = 1.1$, $T_d = 0$, which gives a good response with the nominal flow rate $Q = 1$. It is obvious that the response of the closed loop system is different, when the flow rate changes. With small flow rates the performance deteriorates clearly, because the time delay and the time constant of the process are larger.

In Fig. 6.3 results have been presented, when the time-varying PID$_z$ controller of (6.2) is used. The constant tuning parameters $K_{pz} = 0.5$, $K_{iz} = 0.45$, $K_{dz} = 0$ correspond to the ones in the previous figure. The time-varying controller compensates clearly the effects caused by the different flow rates. The responses are clearly better than those shown in the previous figure.

It is interesting to note that almost identical results as in Figs. 6.2, 6.3 have been obtained in Åström and Wittenmark (1995), where a controller with varying sampling interval has been used as discussed earlier. Possible small deviations are due to the fact that the controller and its parameters are not totally specified in the reference.

The stability of the closed loop system can be studied by means of the theory developed in Section 3.2. The control algorithm (6.2) is equivalent to (6.1), which has constant

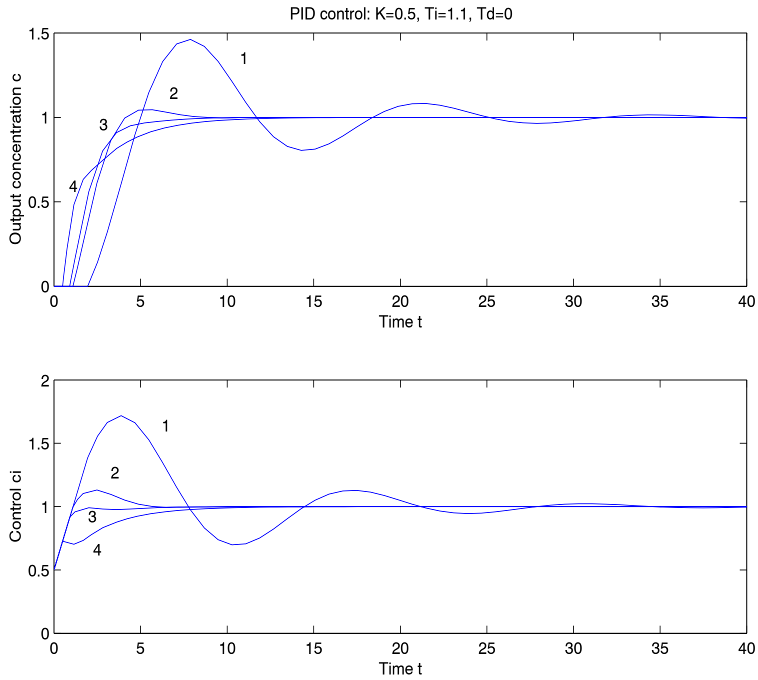

Figure 6.2: Closed loop step responses. PID controller with constant coefficients. 1:Q=0.5, 2:Q=0.9, 3:Q=1.1, 4:Q=2.0.

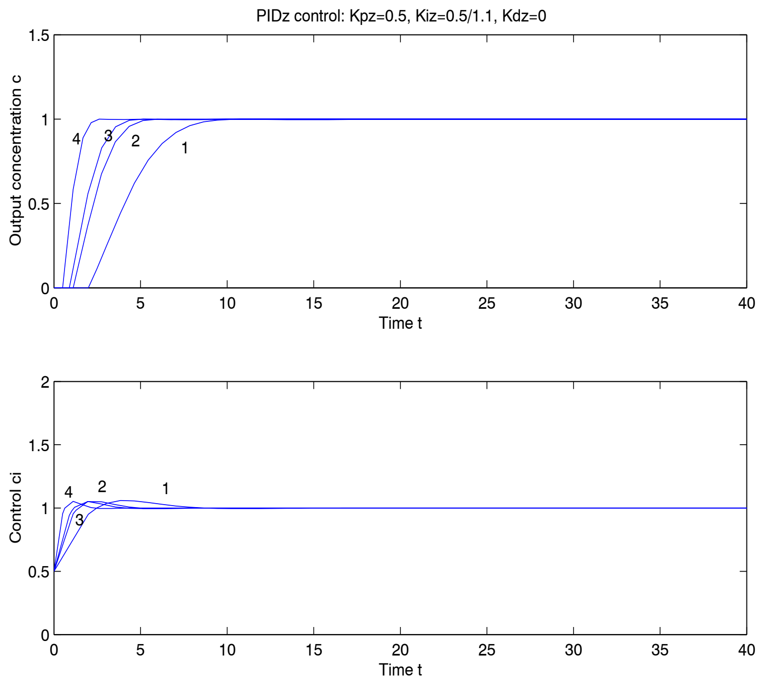

Figure 6.3: Closed loop responses: PID controller with time-variable coefficients. 1:Q=0.5, 2:Q=0.9, 3:Q=1.1, 4:Q=2.0.

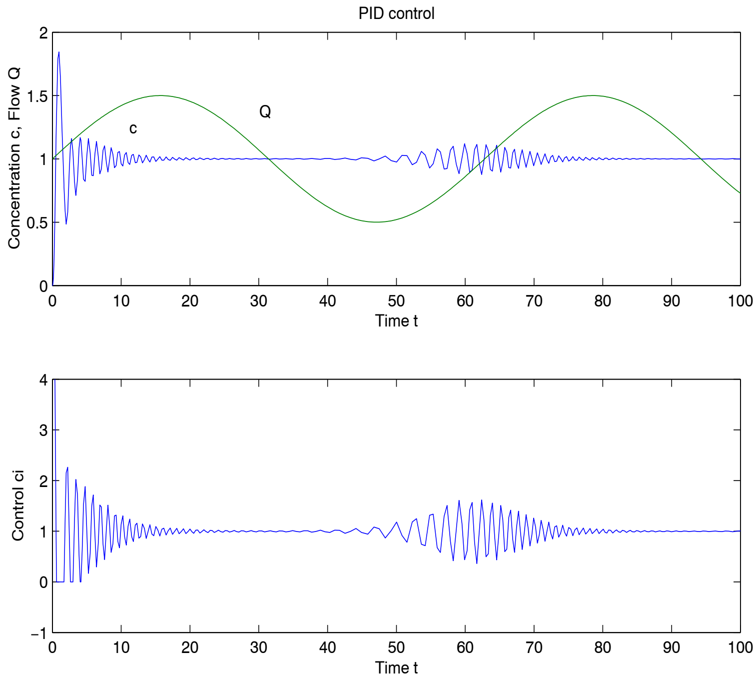

Figure 6.4: PID controller with constant coefficients. K=5, T_i=0.91, T_d=0.

coefficients. If the process model is z-invariant, the closed loop system can thus be modelled in a constant-coefficient form in z-domain. Under the assumption that z tends to infinity as the time variable tends to infinity, the stability of the closed loop system can be determined according to the Proposition 2. Note that the assumption made on z is not a severe limitation, because the flow rate is assumed to be positive for all time instants.

The result that the stability of the closed loop system can be guaranteed in spite of flow variations by using a time-varying controller is an additional motivation to the use of the PID$_z$ controller. As an example, consider the process of three perfect mixers in series under variable flow but constant volumes. Let the process be controlled first by an ordinary PID controller, which is tuned such that for a constant nominal flow rate the closed-loop system is stable but near the stability limit. Next, let the flow rate vary sinusoidally as shown in Fig. 6.4. In the figure, Q is the flow rate, c_i is the controller output, and c is the concentration at the process output. A step change from 0 to 1 has occurred in the reference value in the beginning of the simulations. The numerical values used in the simulations are: $V_1 = V_2 = V_3 = 1/3$, $K_{pz} = 5$, $K_{iz} = 5.5$, $K_{dz} = 0$, $Q_0 = 1$ (in tuning), $Q(t) = 1 + 0.5 \sin(0.1t)$.

The system seems to be stable for large flow rates but starts to oscillate when the flow rate decreases. Loosely speaking, the system thus 'oscillates' between stable and unstable behaviour. The figure Fig. 6.5 shows the same simulation, but the corresponding time-varying PID_z-controller has now been used. The system is stable in spite of flow variations. Notice that if the controller were tuned such that the closed-loop system were

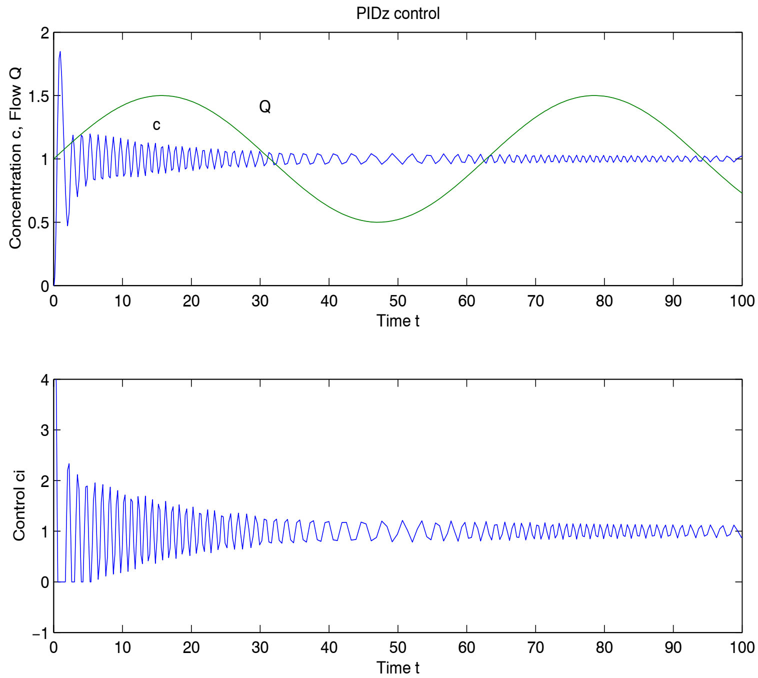

Figure 6.5: PID_z controller. K_{pz}=5, K_{iz}=5.5, K_{dz}=0.

unstable in z-domain, it would then be unstable in time domain for all flow rates.

The variation of the closed loop poles has been presented in Fig. 6.6 for the constant flow rates 0.05, 0.1, 0.5, 1, 1.5, 2 and 2.5. It is seen that the closed-loop system controlled by a PID_z controller is always stable, while the PID controller leads to an unstable system for small flow rates.

6.2 Flow rate as the control signal

In the examples of the previous section the input signal to the process was considered to be a concentration, which was formed by combining the process flow and the reagent flow in a small vessel with no dynamic properties. However, the actual control signal is the reagent flow rate through the control actuator and not concentration directly. In the current section this issue is considered in detail.

Consider the process of an ideally mixed vessel. The process input and output flow rates are $Q_{pi}(t)$ and $Q_{po}(t)$, respectively, and the concentration of a chemical within the process flow is $c_{pi}(t)$. The control signal (flow through a control valve) is $Q_c(t)$, and the reagent concentration is c_c (constant). Usually the reagent is so strong that the control flow rate

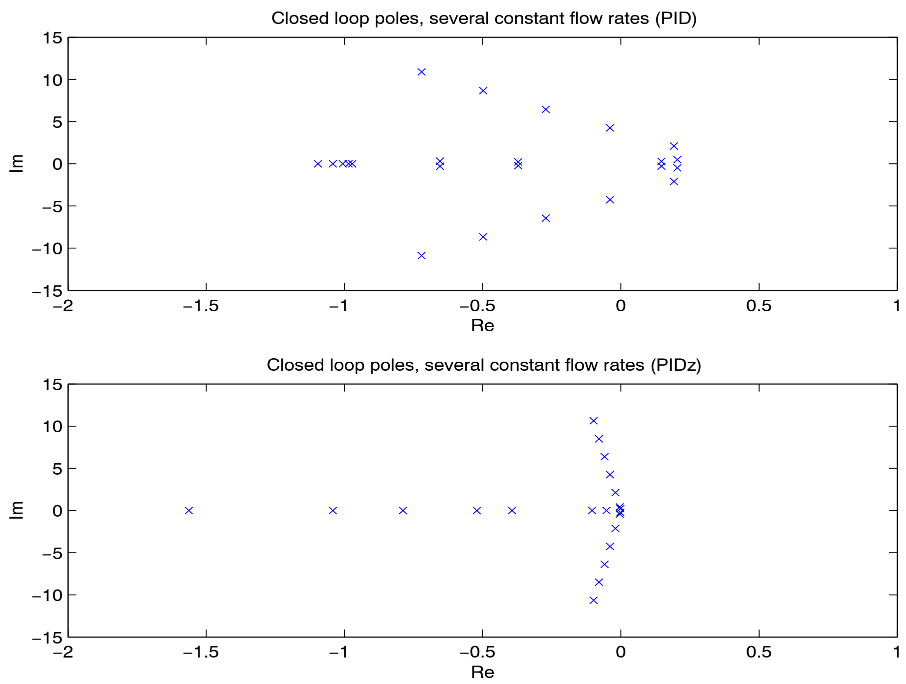

Figure 6.6: Location of closed-loop poles for seven constant flow rates

is very small compared to the process flow rate, $Q_c(t) \ll Q_{pi}(t)$. That implies that if the liquid level is well controlled, then $Q_{pi}(t) \approx Q_{po}(t) \triangleq Q(t)$, and the volume V of the liquid in the vessel is approximately constant. The state equation for the vessel becomes

$$\dot{c}(t) \approx -(Q(t)/V)c(t) + (Q_c(t)/V)c_c + (Q(t)/V)c_{pi}(t) \tag{6.29}$$

where $c(t)$ is the state variable, $Q_c(t)$ is the control signal, and $c_{pi}(t)$ can be regarded as a disturbance term.

The problem with the above equation is that it is not z-invariant, because it is not possible to define a scale z such that the equation would have constant coefficients with respect to z. That is because the ratio between the flow rates $Q(t)$ and $Q_c(t)$ is not necessarily constant.

But consider Fig. 6.1, where the two input flows are first combined in a small vessel (ignore the delay, which is not important now). The idea is that the input to the ideally mixed vessel can be described by the flow rate $Q_i(t)$ and concentration $c_i(t)$. The mixing in the small vessel is assumed to have no dynamics so that it can be described by the following two equations

$$Q_{pi}(t) + Q_c(t) = Q_i(t) \tag{6.30}$$

$$Q_{pi}(t)c_{pi}(t) + Q_c(t)c_c = Q_i(t)c_i(t) \tag{6.31}$$

The input concentration can be solved and written in the form

$$c_i(t) = \frac{1}{1 + Q_c(t)/Q_{pi}(t)} c_{pi}(t) + \frac{Q_c(t)}{Q_{pi}(t) + Q_c(t)} c_c \approx c_{pi}(t) + \frac{Q_c(t)}{Q(t)} c_c \tag{6.32}$$

where the approximations regarding the relations of the flow rates have been taken into account. The equation obtained gives a relationship between the control signal $Q_c(t)$ and the input concentration of the process $c_{pi}(t)$. The state equation of the ideally mixed vessel is

$$\dot{c}(t) = \frac{Q_i(t)}{V}(c_i(t) - c(t)) \approx \frac{Q(t)}{V}(c_i(t) - c(t)) \tag{6.33}$$

Substituting the expression for $c_i(t)$ into this gives equation (6.29), which shows that under the approximations made the idea of mixing the two flow rates in a vessel without dynamics is justified.

To summarize, the process can be modelled by the equations

$$V\dot{c}(t) = -Q(t)c(t) + Q(t)c_i(t) \tag{6.34}$$

$$c_i(t) = c_{pi}(t) + \frac{Q_c(t)}{Q(t)} c_c \tag{6.35}$$

The former expression is the familiar model of an ideally mixed vessel. Although the latter expression can in z-domain be written as

$$\bar{c}_i(z) = \bar{c}_{pi}(z) + \frac{\bar{Q}_c(z)}{\bar{Q}(z)} c_c$$

the equation is dependent on the flow rate, which means that the system model of the two equations is not invariant with respect to z.

The control design can now be based on the fact that the process model is given by a z-invariant differential equation (6.34) and by an algebraic equation (6.35). The dynamic part is controlled by the time-varying PID controller, equation (6.2), and the controller output $c_i(t)$ is realized by choosing $Q_c(t)$ according to (6.35). The actual control signal $Q_c(t)$ thus becomes

$$\begin{aligned} Q_c(t) &= \frac{Q(t)}{c_c}(c_i(t) - c_{pi}(t)) \\ &= \frac{Q(t)}{c_c}(K_{pz}e(t) + K_{iz}\int_{t_0}^{t} \frac{Q(\nu}{V(\nu)})e(\nu)d\nu \\ &\quad + K_{dz}\frac{V(t)}{Q(t)}\frac{de(t)}{dt} - c_{pi}(t)) \end{aligned} \tag{6.36}$$

for $Q_c(t) \geq 0$. The algorithm is a combination of feedback and feedforward control, because the term $c_{pi}(t)$ must be measured from the process flow. If it is not believed to vary much, it is possible to use the nominal value c_{pi0}, which leads to a constant bias term in the algorithm.

In some cases the bias term in the above control algorithm can also be presented in a more attractive way. To this end, use the normal approximation $Q_{pi}(t) \approx Q_i(t) = Q(t)$ and consider small changes in the process and reagent flow rates $Q(t) = Q_o + \Delta Q(t)$, $Q_c(t) = Q_{co} + \Delta Q_{co}(t)$, in which the constants Q_o and Q_{co} denote nominal flows. The controller (6.36) can be written as

$$Q_c(t) = \frac{Q(t)}{Q_o}[\frac{K_{pz}Q_o}{c_c}e(t) + \frac{K_{iz}Q_o}{c_c}\int_{t_0}^{t}\frac{Q(\nu)}{V(\nu)}e(\nu)d\nu + \frac{K_{dz}Q_o}{c_c}\frac{V(t)}{Q(t)}\frac{de(t)}{dt} - \frac{Q_o}{c_c}c_{pi}(t)] \quad (6.37)$$

In a pH-process, where strong base (process flow) is neutralized by a strong acid (reagent) it holds in steady state (pH=7) that $Q_0 c_{pi} = -Q_{c0}c_c$. This can be explained by noticing that a strong acid can be regarded as a negative base, and the goal in control is to keep the value c_i close to zero. The two input flows thus form a neutral salt. It then follows that

$$\begin{aligned} Q_c(t) &= \frac{Q(t)}{c_c}(c_i(t) - c_{pi}(t)) \\ &= \frac{Q(t)}{Q_o}(K_p e(t) + K_i\int_{t_0}^{t}\frac{Q(\nu}{V(\nu)})e(\nu)d\nu \\ &\quad + K_d\frac{V(t)}{Q(t)}\frac{de(t)}{dt} + Q_{co}) \end{aligned} \quad (6.38)$$

in which $K_p = K_{pz}Q_o/c_c$, $K_i = K_{iz}Q_o/c_c$ and $K_d = K_{dz}Q_o/c_c$. The algorithm was invented by Jutila, see e.g. (Jutila and Jaakola, 1986), who derived the equation using heuristic methods. Actually, Jutila's algorithm uses slightly different notations, and it is most often written in discrete time; however, it is identical to the equation presented above. It is interesting to note that the algorithm can be derived starting from the z-transformation technique as described.

Finally, some remarks about the approximations used in the above derivations are made. The practical but difficult problem of pH control has been discussed widely in the literature, and several algorithms have been tested. In one example discussed by Jutila (1983) a waste-water treatment plant was considered. In principle, the system can be described as a similar concentration process as described above. Some typical values of the variables in the plant are $Q_{pi} = 1\,\mathrm{m}^3/\mathrm{s}$, $c_{pi} = 5\,\mathrm{mol/m}^3$, $Q_c = 0.0001\,\mathrm{m}^3/\mathrm{s}$, $c_c = 15000\,\mathrm{mol/m}^3$. Clearly $Q_c \ll Q_{pi}$, which leads to $Q_c/Q_{pi} = 0.0001$ and $Q_{pi} + Q_c = 1.0001\,\mathrm{m}^3/\mathrm{s} \approx 1\,\mathrm{m}^3/\mathrm{s}$. Substituting the numerical values into the equation (6.32) gives $c_i = (4.9995 + 1.5/1.0001)\,\mathrm{mol/m}^3 \approx (5 + 1.5)\,\mathrm{mol/m}^3$. The approximations made in (6.32) are thus found to be good in the particular process example.

The same testing equipment, which was described in Section 5.1, was again used to test the PID type controllers. The purpose of these tests was to compare a conventional and

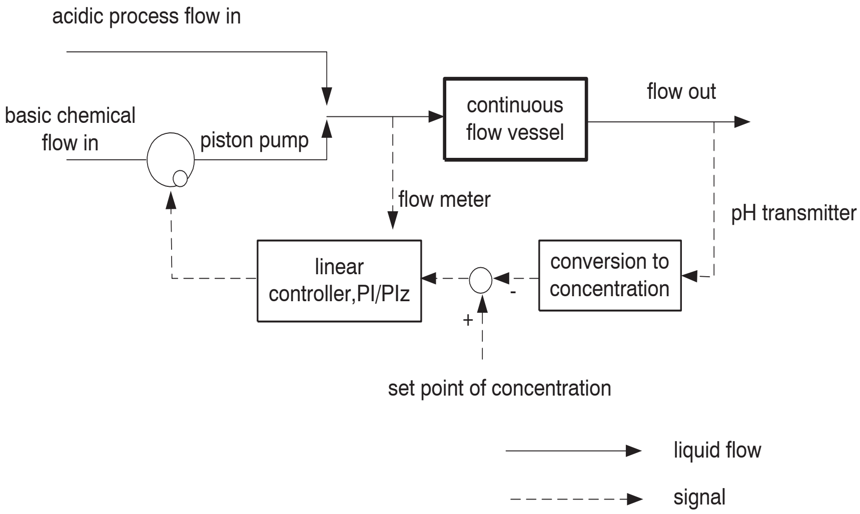

Figure 6.7: Scheme of the pilot-plant

a new controller with each other, and especially to verify the theoretical results on the stability aspects of the corresponding control loops. A diagram of the test system is shown in Fig. 6.7.

The testing equipment (pilot-plant) has actually been constructed to control the acidity of liquid in the vessel. This is the well-known pH control problem, which is nonlinear in nature and which has been discussed widely in the literature. In the case of strong acids and strong bases it is possible to linearize the control loop by the titration curve, which means that the problem is approximated by a linear concentration control problem. The approximation is good as long as the assumption of strong acids and strong bases is valid. For details, see (Niemi and Jutila, 1977b), (Jutila, 1983), (Jutila and Jaakola, 1986), (Jutila *et al.*, 1999).

As the process flow passed continuously the process vessel, the output pH was measured and converted to concentration by using the stored titration curve. The setpoint of acidity was fixed to a constant value in the beginning of a test. The controllers were used to regulate the small amount of strong chemical reagent by means of a controllable pump. The process flow was held at a constant value and at a chosen time changed stepwise to another constant value.

Both controllers were tuned to give a similar, stable behaviour of the system at the initial part of the test, when the flow rate was high. Damped oscillations were observed at the output as this approached the reference value. In the second phase after the flow rate had been reduced to 50 % of the original value, the loop with the PID controller with constant coefficients became unstable exhibiting a growing oscillation up to saturation (see Fig. 6.8). The loop with PID_z controller remained stable and showed a similar behaviour

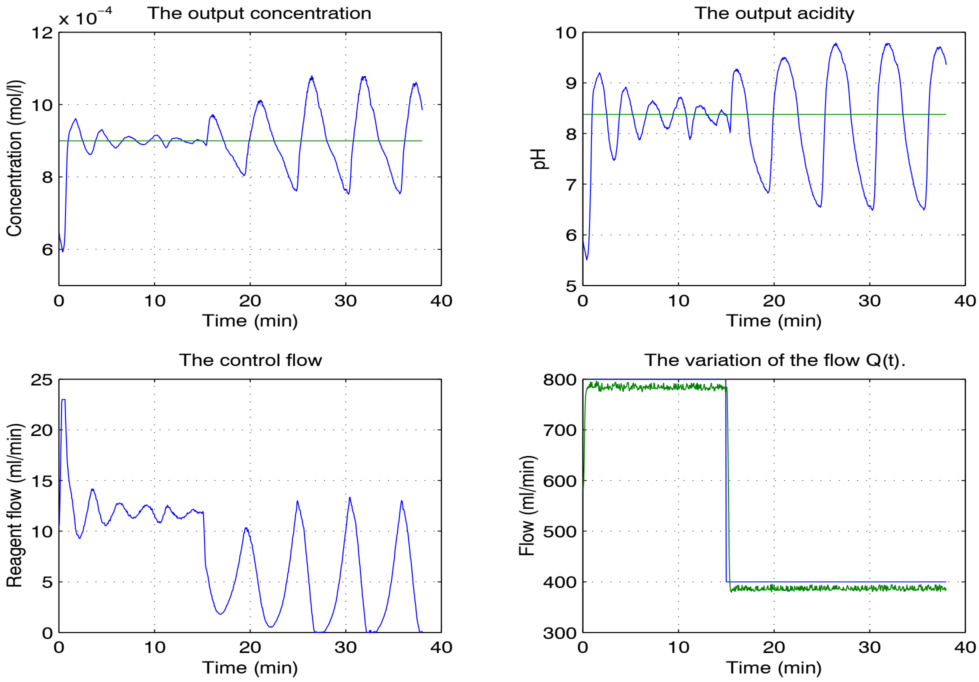

Figure 6.8: Test 1: PID controller with constant coefficients

in both parts of the test (see Fig. 6.9). The result agrees completely with the theory.

Numerical values related to the performed tests were: Process liquid, HCL in water, pH≈ 3.3; Control liquid, NaOH in water, pH≈ 12.1; Liquid volume in the vessel 0.7l; Process flow rate 800 ml/min, 400 ml/min; Chemical flow 0-23 ml/min. In the beginning of both tests the set point of concentration has been changed from 7E-4 mol/l (pH≈ 5.8) to 9E-4 (pH≈ 8.3).

To end this section a few general comments are appropriate. The control problem of a mixing tank discussed is both nonlinear and time-varying. Nonlinearity is caused by the fact that the control signal is reagent flow rate, not concentration, whereas time variability is caused by the changing process flow rate. The solution proposed was to divide the process equations into static and dynamic parts thus making it possible to use the modified time scale to simplify the time-varying dynamics, and the static part to generate the final control signal (reagent flow rate). Although the solution is neat, it required the assumption of a small reagent flow rate compared to the process flow rate. In the first test a discretized version of a conventional PID controller was used (programmed in PC); the time-variable PID_z controller was used in the second test, and the results were compared. If the mentioned approximation is not valid, it might be possible to use some other kind of linearization technique. For some literature on this issue, see e.g. (Balchen *et al.*, 1988), (Henson and Seborg, 1990) and (Kravaris and Soroush, 1990). These papers discuss methods to use suitable variable transformations to make the system linear in

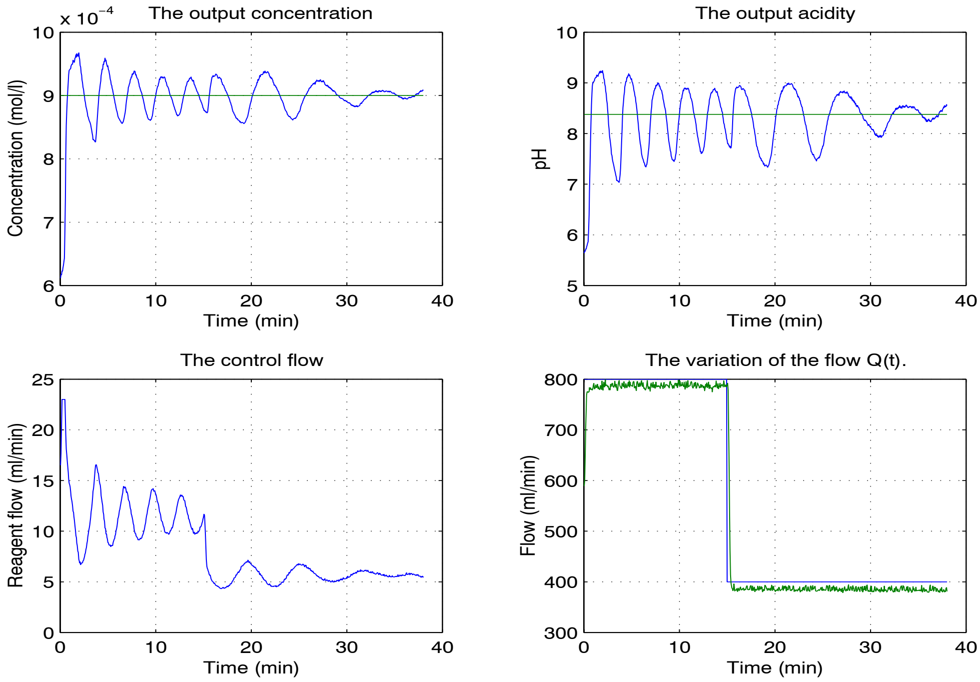

Figure 6.9: Test 2: PID controller with time-variable coefficients (PID$_z$-controller)

terms of the new variables, whereafter a linear controller operating on the new variables is designed. A nice application in the multivariable nonlinear control problem of a mixing tank using these ideas has been reported by Häggblom (1993). However, phenomena related to time-varying processes were not discussed in the mentioned papers. Also, the variable transformations do not include any modification of the time scale.

From a more general system theoretic viewpoint there exist a wide literature on linearization techniques and their applicability in controller design. For a good textbook see e.g. (Marino and Tomei, 1995).

6.3 LQ optimal control

Besides PID control it is reasonable to consider other control strategies, which are based on the structural model of the process. Among the classical control strategies is the *linear quadratic (LQ) control*, which is based on the idea that the feedback law is determined in order to minimize a given quadratic cost function.

Consider the process model

$$\dot{x}(t) = A(t)x(t) + B(t)u(t) \qquad x(t_0) = x_0 \tag{6.39}$$

$$y(t) \quad = \quad C(t)x(t) + D(t)u(t)$$

where the dimensions of the functions are as given in Section 3.1. The cost criterion to be minimized is

$$J(t_0) = \frac{1}{2}x^T(t_f)S(t_f)x(t_f) + \frac{1}{2}\int_{t_0}^{t_f}[x^T(t)X(t)x(t) + u^T(t)R(t)u(t)]dt \tag{6.40}$$

where $t_0..t_f$ is the optimization horizon and $S(t_f)$, $X(\cdot)$ and $R(\cdot)$ are suitable $n \times n$, $n \times n$ and $m \times m$ dimensional weighting matrices. $S(t_f)$ is assumed to be positive semidefinite; $X(t)$ and $R(t)$ are assumed to be positive semidefinite and positive definite, respectively.

It is well known, see e.g. (Lewis and Syrmos, 1995), that the solution to the problem is the state feedback law

$$u^*(t) = -R^{-1}(t)B^T(t)S(t)x(t) \tag{6.41}$$

where the function $S(\cdot)$ is given as a solution to the *Riccati equation*

$$-\dot{S}(t) = A^T(t)S(t) + S(t)A(t) - S(t)B(t)R^{-1}(t)B^T(t)S(t) + X(t) \tag{6.42}$$

with the end condition $S(t_f)$. The minimum cost achieved by using this control law is

$$J^*(t_0) = \frac{1}{2}x^T(t_0)S(t_0)x(t_0) \tag{6.43}$$

Note that the above control problem is a *regulator problem*, in which it is desirable to drive the state close to the origin of the state space. If the objective is to follow a reference trajectory, a *servo* or *tracking* problem has to be considered. However, in most control problems in the process industry the objective is to keep the controlled variable constant. In that case the state equations can always be scaled so that the desired state is zero.

Even though the solution to the control problem seems to be compact, it is difficult to solve it in practice, because the Riccati equation is a group of nonlinear differential equations with time-varying coefficients. If the optimization horizon is long ($t_f - t_0$ large) the suboptimal solution can be used in the time-invariant case (Lewis and Syrmos, 1995), (Anderson and Moore, 1989), setting $\dot{S}(t) = 0$ and hence using the stationary solution of the Riccati equation. For time-invariant processes this is even desirable, because it leads to a state feedback law, in which the controller gain matrix is constant. Moreover, good software is available to solve the problem numerically. The situation is not the same in the case of time-varying processes, because the Riccati equation still has time-varying coefficients and is therefore difficult to solve.

But consider a process model, in which $A(t) = k(t)\bar{A}$ and $B(t) = k(t)\bar{B}$. The system is z-invariant with respect to the scale

$$z = f(t) = \int_{t_0}^{t} k(\nu)d\nu \tag{6.44}$$

If the weighting matrices in the cost criterion are chosen as $X(t) = k(t)\bar{X}$, $R(t) = k(t)\bar{R}$, it follows that the system equations and cost criterion change into the form

$$\frac{d\bar{x}(z)}{dz} = \bar{A}\bar{x}(z) + \bar{B}\bar{u}(z) \tag{6.45}$$

$$\bar{J}(0) = \frac{1}{2}\bar{x}^T(z_f)\bar{S}(z_f)\bar{x}(z_f) + \frac{1}{2}\int_0^{z_f} [\bar{x}^T(z)\bar{X}\bar{x}(z) + \bar{u}^T(z)\bar{R}\bar{u}(z)]dz \tag{6.46}$$

such that for all t and the corresponding z, $\bar{S}(z_f) = S(t_f)$, $\bar{x}(z) = x(t)$, $\bar{u}(z) = u(t)$. The problem has changed into a 'time-invariant' form (with respect to z), and the solution is given by the equation

$$-\frac{d\bar{S}}{dz}(z) = \bar{A}^T\bar{S}(z) + \bar{S}(z)\bar{A} - \bar{S}(z)\bar{B}\bar{R}^{-1}\bar{B}^T\bar{S}(z) + \bar{X} \tag{6.47}$$

with the given end condition $\bar{S}(z_f)$, and

$$\bar{u}^*(z) = -\bar{R}^{-1}\bar{B}^T\bar{S}(z)\bar{x}(z) \tag{6.48}$$

The minimum cost is

$$\bar{J}^*(0) = \frac{1}{2}\bar{x}^T(0)\bar{S}(0)\bar{x}(0) \tag{6.49}$$

Note that in time and z-domains the solutions to the Riccati equations, the resulting control laws and the system dynamics are equivalent. Additionally, if the stationary solution for $\bar{S}$ is used, the feedback law becomes in time domain

$$u^*(t) = -\bar{R}^{-1}\bar{B}^T\bar{S}x(t) \tag{6.50}$$

which can be directly used.

However, it should be noticed that the above equivalence holds only, if the weighting matrices in the cost criterion are chosen as $X(t) = k(t)\bar{X}$, $R(t) = k(t)\bar{R}$. For other choices the problem still remains 'time-varying' even in z-domain, and the use of the new scale does not make the solution simpler.

Consider the example presented earlier in Section 6.1, in which an unstable closed loop system was stabilized by a PID_z controller. If Fig. 6.10 simulation results have been presented, in which PID_z control and LQ control have been applied. The simulation time interval has now been 0..10 (earlier 0..100) to investigate the transient response in more detail. In the beginning the concentration reference has been changed from zero to one, and from the time $t = 5$ to $t = 6$ a pulse disturbance of amplitude 0.3 has affected the input of the process. In LQ control the controller has the form

$$u^*(t) = -Lx(t) + c_k r(t) \tag{6.51}$$

where $r(t)$ is the reference and the constant c_k is chosen such that the static gain from the concentration reference to output concentration is one. In all curves the solid line denotes

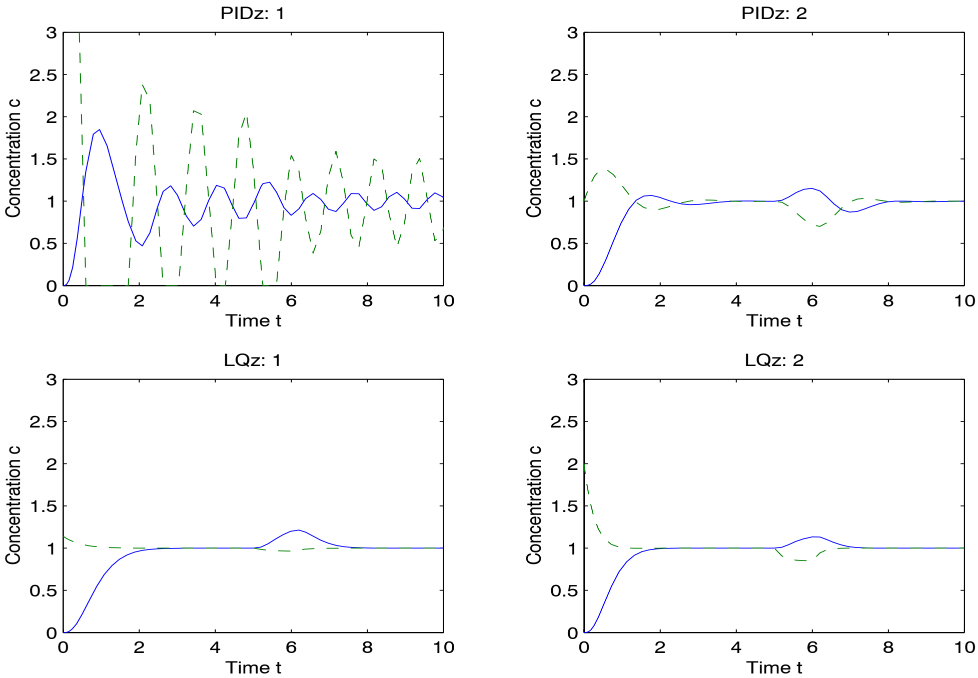

Figure 6.10: Concentration control by PID_z and LQ_z controllers

the output concentration, while the dashed line is the control signal. In the upper left plot the PID_z controller with the same tuning as earlier ($K_{pz} = 5$, $K_{iz} = 5.5$, $K_{dz} = 0$) has been used. A stable but oscillatory response resulted, as expected. Result obtained by using a better tuning ($K_{pz} = 1$, $K_{iz} = 1.3$, $K_{dz} = 0$) is shown in the upper right plot. That result is clearly comparable to the two lower plots, where LQ control was applied. The tuning matrices in the lower left and right plots were

$$\bar{X} = \begin{bmatrix} 0.01 & 0 & 0 \\ 0 & 0.01 & 0 \\ 0 & 0 & 0.01 \end{bmatrix}, \ \bar{R} = 0.1$$

$$\bar{X} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \ \bar{R} = 1$$

respectively.

All previous controllers can naturally be understood as pole-placement algorithms, and differences in the closed loop behaviour can naturally be explained in this way. The poor PID_z tuning was because the closed loop had a poorly damped pair of poles -0.04+j4.3, -0.04-j4.3. A better tuning moved that pair to -1.1+j2.2, -1.1-j2.2. In the LQ case the oscillatory poles were at the locations -2.8+j0.7, -2.8-j0.7 and -3.3+j1.4, -3.3-j1.4, respectively.

6.4 State feedback and state observer

A classical design method for systems represented by state-space representations is *pole-placement*, in which the concepts of *state feedback* and *state observer* are utilized (see any classical textbook of control, e.g. (Chen, 1999)). Consider the system representation (6.39) with $A(t) = k(t)\bar{A}$, $B(t) = k(t)\bar{B}$, $C(t) = \bar{C}$ and $D(t) \equiv 0$ (for simplicity). In general, the state feedback is formed by

$$u(t) = -Lx(t) + r(t) \tag{6.52}$$

in which L is the feedback coefficient matrix and r is the reference signal, which possibly includes a pre-filter to make the static gain from the reference to the system output to the value one. The state equation in closed loop becomes

$$\dot{x}(t) = k(t)(\bar{A} - \bar{B}L)x(t) + k(t)\bar{B}r(t) \tag{6.53}$$

which is z-invariant. The result can be well understood in the light of the previous section, in which the weights in the cost function were chosen in a way that leads to a state feedback control law with a constant coefficient matrix.

It is interesting to note that the 'frozen' eigenvalues of the closed loop system matrix $k(t)(\bar{A} - \bar{B}L)$ change with $k(t)$ along straight lines. To explain, consider $k_t = k(t)$ at an arbitrary but fixed time t. The characteristic equation at this time instant is

$$p_t(s) = \det[sI - k_t(\bar{A} - \bar{B}L)] = \det[k_t(\frac{s}{k_t}I - \bar{A} + \bar{B}L)] = \det(k_t)\det(\bar{s}I - \bar{A} + \bar{B}L) = 0 \tag{6.54}$$

in which $\bar{s} = s/k_t$. Because $k(t)$ is a positive function, the 'frozen' poles are seen to vary along straight lines in the complex plane. If the pole-placement design is carried out in z-domain, the poles are placed in any chosen fixed places in the complex plane. With respect to ordinary time domain the same controller (6.52) leads to a time-varying closed-loop system, in which the 'frozen' eigenvalues change along straight lines in the complex plane. (This characteristics was clearly visible earlier in Fig. 6.6.)

By using the analogy to time-invariant systems, it is noticed that the damping ratio remains constant but the natural frequency changes, as k_t varies. That means that the oscillation amplitude remains the same, but the frequency of the oscillation changes. That makes sense, because the operation in time and z-domains involve a time scaling.

In general, the transfer function at one specified time instant is

$$\begin{aligned} G_t(s) &= \bar{C}(sI - k_t\bar{A})^{-1}k_t\bar{B} \\ &= \bar{C}\left[k_t(\tfrac{s}{k_t}I - \bar{A})\right]^{-1} k_t\bar{B} \\ &= \bar{C}(\tfrac{s}{k_t}I - \bar{A})^{-1}\bar{B} \\ &= \bar{C}(\bar{s}I - \bar{A})^{-1}\bar{B} \end{aligned} \tag{6.55}$$

where $\bar{s} = s/k_t$. The result shows that the poles and zeros of a system in z-domain $(\bar{A}, \bar{B}, \bar{C})$ correspond to those of the 'frozen' system at time t. In this respect, possible pole-zero cancellations occur in both cases, too.

However, it must be kept in mind that the analysis of time-varying systems pointwise in time by classical methods is not scientifically sound. For example, the frozen eigenvalues of a time-varying system do not give information on system stability, see e.g. (Rugh, 1993). However, the application of the Proposition 2 shows that a z-invariant system is stable, if the 'frozen' eigenvalues remain in the left half plane in spite of changes in $k(t)$.

The state observer can be presented in the form

$$\begin{aligned} \frac{d\hat{\bar{x}}(z)}{dz} &= \bar{A}\hat{\bar{x}}(z) + \bar{B}\bar{u}(z) + K\left[\bar{y}(z) - \bar{C}\hat{\bar{x}}(z)\right] \\ \hat{\bar{y}}(z) &= \bar{C}\hat{\bar{x}}(z) \end{aligned} \tag{6.56}$$

where the ˆ notation is used to mean an estimate value. Using (6.44) the equation is in time domain

$$\dot{\hat{x}}(t) = k(t)\bar{A}\hat{x}(t) + k(t)\bar{B}u(t) + k(t)K\left[y(t) - \bar{C}\hat{x}(t)\right] \tag{6.57}$$

and

$$\dot{\hat{x}}(t) = k(t)\left(\bar{A} - K\bar{C}\right)\hat{x}(t) + k(t)\begin{bmatrix} \bar{B} & K \end{bmatrix}\begin{bmatrix} u(t) \\ y(t) \end{bmatrix} \tag{6.58}$$

which is z-invariant. The dynamics of the estimation error

$$\tilde{x}(t) = x(t) - \hat{x}(t) \tag{6.59}$$

becomes

$$\dot{\tilde{x}}(t) = k(t)\left[\bar{A} - K\bar{C}\right]\tilde{x}(t) \tag{6.60}$$

As expected, the dynamics of the estimation error can be described by the eigenvalues (calculated pointwise in time) of the estimator, which change as multiples of $k_t = k(t)$. With respect to the z-scale, the eigenvalues are constant.

If the estimated state is used in feedback as

$$u(t) = -L\hat{x}(t) + r(t) \tag{6.61}$$

the closed loop equation can be written as

$$\dot{x}(t) = k(t)(\bar{A} - \bar{B}L)x(t) + k(t)\bar{B}L\tilde{x}(t) + k(t)\bar{B}r(t) \tag{6.62}$$

The dynamic equations for the state and state error can be combined leading to the representation

$$\begin{aligned} \begin{bmatrix} \dot{x}(t) \\ \dot{\tilde{x}}(t) \end{bmatrix} &= k(t)\begin{bmatrix} (\bar{A} - \bar{B}L) & \bar{B}L \\ 0 & (\bar{A} - K\bar{C}) \end{bmatrix}\begin{bmatrix} x(t) \\ \tilde{x}(t) \end{bmatrix} + k(t)\begin{bmatrix} \bar{B} \\ 0 \end{bmatrix} r(t) \\ y(t) &= \begin{bmatrix} \bar{C} & 0 \end{bmatrix}\begin{bmatrix} x(t) \\ \tilde{x}(t) \end{bmatrix} \end{aligned} \tag{6.63}$$

which shows that the classical results on the separation of state feedback design and observer design are analogously valid for z-invariant representations.

Chapter 7

Z-invariant Systems

The concept of z-invariant state-space representation was introduced in Chapter 3. The conditions obtained for a representation to be z-invariant are generally restrictive, although they are satisfied for representations of certain system classes like simple flow process models with varying flow rate. The purpose of the current chapter is to consider the concept of a z-invariant system a little more deeply than in the earlier practically oriented chapters. Also, it is then natural to consider alternative approaches for the analysis and controller design of such time-varying processes, which are not z-invariant.

7.1 Preliminary definitions and concepts

The concept of a z-invariant system is a natural extension to that of a time-invariant system. Some basic definitions are now given, so that the presentation and analysis of z-invariant systems becomes possible. The following definitions and results are based on the material given in (Zadeh and Desoer, 1963) and (Padulo and Arbib, 1974).

Let a system Σ be defined as an input-output mapping, $S(t, \tau, x, u)$, which has the property that the *state* x of the system at a certain time instant τ and the *input* u from that time on uniquely determine the *output* of the system for all times $t > \tau$. The input, output and state spaces as well as the structure of the mapping have been chosen properly to meet some basic consistency conditions (Zadeh and Desoer, 1963). Moreover, a unique zero state 0 is known to exist, which produces zero output, when zero input is applied. Using an arbitrary input acting on a system originally at zero state, the *zero state response* $S(t, \tau, 0, u)$ is obtained. When zero input is applied to the system being in an arbitrary state, the *zero input response* $S(t, \tau, x, 0)$ of the system is produced.

The system is *zero state time-invariant*, if for all inputs and all time shifts, the zero state response of the system shifted in time is identical to the zero state response of the system, when the input has been shifted correspondingly. The weighting function can only assure that the system in question is zero state time-invariant, but it is impossible to deduce whether the system is generally time-invariant.

The system is called *zero input time-invariant*, if for all initial states α, all initial times t_0, and all time shifts δ the zero input response of the system starting in state α at time $t_0 - \delta$ is identical (to within a translation by amount δ along the time axis) with the zero input response of the system starting in state α at time t_0.

Finally, the system is (generally) time-invariant, if for all initial times t_0 and starting states α, all inputs u, and all shifts δ the output, when translated δ units in time, is equal to the output produced by using the translated input to the system being in state α. If the condition given holds for certain initial states only, the system is called time-invariant with respect to these states.

In addition to the above definitions it is possible to use the concept of a *weakly time-invariant system*. This property means that if (u, y) is the input-output pair of the system, so is the pair, in which both u and y are shifted in time by an arbitrary amount.

If a system is weakly time-invariant, it is not necessarily time-invariant generally, but for a linear differential system the implication holds. More important to the current text is the result that if a linear system is both zero state and zero input time-invariant, it is time-invariant. The above result can then be stated mathematically as follows: A linear system is time-invariant, if and only if its input-output relation admits the representation

$$y(t) = F(t - t_0)x(t_0) + \int_{t_0}^{t} p'(t - \tau, 0)u(\tau)d\tau \tag{7.1}$$

for all t_0, $x(t_0)$ and u. If the system representation can be written as above for certain initial states $x(t_0)$ only, the system is time-invariant with respect to these states.

Next, consider two systems Σ and Σ_1, which have the same set of admissible input functions Ω. The state x of Σ is defined to be equivalent to the state x_1 of Σ_1 at time τ, if for any admissible input the following holds as $t > \tau$:

$$S_\Sigma(t, \tau, x, u) = S_{\Sigma_1}(t, \tau, x_1, u)$$

The condition simply means that the states are equivalent if, when using the same input, the same output results for both systems (the number of inputs as well as the number of outputs of the systems are assumed to be the same).

Two systems Σ and Σ_1 are *equivalent* if, for any time t_0 and for every state x of Σ there is a state x_1 of Σ_1, which is equivalent to x at t_0, and vice versa (for any state of Σ_1 there is an equivalent state in Σ).

Two systems Σ and Σ_1 are *zero state equivalent*, if for every zero state 0 of Σ there is a zero state 0_1 of Σ_1 such that

$$S_\Sigma(t, t_0, 0, u) = S_{\Sigma_1}(t, t_0, 0_1, u)$$

for every input function u.

Note that in the current text it is always assumed that the zero state of the system is the origin of the state space. Assuming that the two systems have equal dimensions it would be possible to write 0 instead of 0_1 in the definition above. The zero state is defined in the same way as before; it is a state of the system, which produces zero output for all $t > t_0$, when the system is in that state at time t_0 and zero input is used from there on.

Two systems that have identical zero state responses are zero state equivalent, and in linear case zero state equivalent systems have identical impulse responses. If two linear systems are equivalent, they are also zero state equivalent, but the converse is not necessarily true.

To put these ideas into a more practical setting, a more concrete model class must be considered. As discussed earlier, the weighting function provides a convenient input-output model for linear systems, which are relaxed initially. If a state-space realization is available, a rich spectrum of analysis and synthesis methods are available.

For future use consider the following well-known result (Padulo and Arbib, 1974), (Rugh, 1993): If the system is described by its weighting function $p'(t, \tau)$, the system has a state-space realization, if and only if there exist functions $K(\cdot)$ and $L(\cdot)$ such that for all $t \geq \tau$:

$$p'(t, \tau) = K(t)L(\tau)$$

7.2 Linear differential systems

The concept of equivalent systems is a rather general one, because no assumptions about linearity or the dimensions of the state spaces of the systems have been made. Even for linear cases, equivalent systems do not necessarily have the same amount of state variables, which usually makes it difficult to find representations, from which the equivalence can be deduced. There is however one concept, called algebraic equivalence, which presumes state representations of the same dimensions (Padulo and Arbib, 1974). The concept can be motivated as follows. Consider the *original system*

$$\dot{x}(t) = A(t)x(t) + B(t)u(t) \qquad x(t_0) = x_0 \tag{7.2}$$

$$y(t) = C(t)x(t) + D(t)u(t)$$

and a state transformation

$$x(t) = P(t)s(t) \tag{7.3}$$

where $x(t)$ and $s(t)$ are n-dimensional vectors and $P(t)$ is a non-singular $n \times n$ dimensional matrix. Regarding s a new state variable the realization of the *target system* becomes

$$\begin{aligned} \dot{s}(t) &= E(t)s(t) + F(t)u(t) \qquad s(t_0) = s_0 \\ y(t) &= G(t)s(t) + H(t)u(t) \end{aligned} \tag{7.4}$$

where $s_0 = P(t_0)^{-1}x(t_0) = P_0^{-1}x_0$ and

$$\begin{aligned} E(t) &= P^{-1}(t)[A(t)P(t) - \dot{P}(t)] \\ F(t) &= P^{-1}(t)B(t) \\ G(t) &= C(t)P(t) \\ H(t) &= D(t) \end{aligned} \tag{7.5}$$

Two systems represented by (7.2) and (7.4) are called *algebraically equivalent*, which actually means that a non-singular square matrix $P(t)$ can be found such that two system realizations can be obtained from each other by using the given linear state transformation. If a constant matrix P is used, it is called the similarity transformation in classical control literature.

Let us look at algebraically equivalent systems a little closer for the purpose of future use. Considering the autonomous part of the state equations in (7.2) and (7.4), the solutions become $x(t) = \Phi_A(t, t_0)x_0$ and $s(t) = \Phi_E(t, t_0)s_0$, where $\Phi_A(\cdot, \cdot)$ and $\Phi_E(\cdot, \cdot)$ are the state transition matrices of the system matrices $A(\cdot)$ and $E(\cdot)$, respectively. By combining these solutions with the fact that $x(t) = P(t)s(t)$ it follows

$$P(t) = \Phi_A(t, t_0)P_0\Phi_E^{-1}(t, t_0) \tag{7.6}$$

Also, the transformation matrix $P(\cdot)$ can be obtained as a solution to the initial value problem

$$\dot{P}(t) = A(t)P(t) - P(t)E(t), \quad P(t_0) = P_0 \tag{7.7}$$

The relationship of the two state-transition matrices becomes accordingly

$$\Phi_A(t, \tau) = P(t)\Phi_E(t, \tau)P^{-1}(\tau) \tag{7.8}$$

or

$$\Phi_E(t, \tau) = P^{-1}(t)\Phi_A(t, \tau)P(\tau) \tag{7.9}$$

By using these relationships it follows that the weighting functions of the input-output systems become

$$p'_A(t, \tau) = C(t)\Phi_A(t, \tau)B(\tau) \tag{7.10}$$

$$\begin{aligned} p'_E(t,\tau) &= G(t)\Phi_E(t,\tau)F(\tau) \\ &= C(t)P(t)\Phi_E(t,\tau)P^{-1}(\tau)B(\tau) \\ &= C(t)\Phi_A(t,\tau)B(\tau) \end{aligned} \tag{7.11}$$

(for simplicity, it has been assumed that the systems are strictly proper so that $D(t) \equiv 0$ and $H(t) \equiv 0$; for a short discussion, see Section 7.3). The result shows that the weighting functions and impulse responses of algebraically equivalent systems are the same, which is a well-known fact, see e.g. (Tsakalis and Ioannou, 1993).

As for controllability and observability, consider the controllability gramian

$$W_A(t_0,t_1) = \int_{t_0}^{t_1} \Phi_A(t_0,t)B(t)B^T(t)\Phi_A^T(t_0,t)\,dt \tag{7.12}$$

which for the target system becomes

$$\begin{aligned} W_E(t_0,t_1) &= \int_{t_0}^{t_1} \Phi_E(t_0,t)F(t)F^T(t)\Phi_E^T(t_0,t)\,dt \\ &= \int_{t_0}^{t_1} \Big\{P^{-1}(t_0)\Phi_A(t_0,t)P(t)P^{-1}(t)B(t)B^T(t)(P^T(t))^{-1}P^T(t)\cdot \\ &\qquad \Phi_A^T(t_0,t)(P^T(t_0))^{-1}\Big\}\,dt \\ &= P^{-1}(t_0)W_A(t_0,t_1)(P^T(t_0))^{-1} \end{aligned} \tag{7.13}$$

Because the matrix $P(t_0)$ has full rank, the definiteness of the gramians W_A and W_E is the same. Controllability remains thus invariant in the transformation. A similar calculation shows that the observability gramian

$$M_A(t_0,t_1) = \int_{t_0}^{t_1} \Phi_A^T(t,t_0)C^T(t)C(t)\Phi_A(t,t_0)\,dt \tag{7.14}$$

changes into the form

$$M_E(t_0,t_1) = P^T(t_0)M_A(t_0,t_1)P(t_0) \tag{7.15}$$

Observability is invariant with respect to the transformation.

To investigate the preservation of stability, the important concept of a *Lyapunov transformation* is introduced. Results related to this theory can be seen here and there in the literature, see e.g. (Lyapunov, 1966), (Rugh, 1993), (Harris and Miles, 1980), (Markus, 1955), (Nemytskii and Stepanov, 1960). Consider an autonomous system (7.2), where $B(t) \equiv 0$, $C(t) \equiv 0$, $D(t) \equiv 0$. Define the class M_n of $n \times n$ matrix-valued functions $X(\cdot)$, which satisfies the conditions

- the elements of $X(\cdot)$ are bounded and continuous for every t in $\Re$,

- $\dot{X}(\cdot)$ exists, and its entries are bounded and continuous for every t in $\Re$,
- there exists a constant $M > 0$ such that $\mid \det X(t) \mid > M$ for every t in $\Re$.

In the case that the transformation $P(\cdot)$ satisfies the above three conditions, i.e. $P(\cdot) \in M_n$, it is called a Lyapunov transformation. The stability properties are then known to be the same in the (autonomous) original and in the (autonomous) target system (7.4), $F(t) \equiv 0$, $G(t) \equiv 0$, $H(t) \equiv 0$. There are alternative formulations on the stability conditions in literature, but the key issue is to determine, whether the matrix

$$P(t) = \Phi_A(t, t_0) P_0 \Phi_E^{-1}(t, t_0) \tag{7.16}$$

is a Lyapunov-transformation matrix or not. As long as the transition matrices $\Phi_A(\cdot, \cdot)$ and $\Phi_E(\cdot, \cdot)$ are not known, there seems to be no general procedure to determine this.

In the above definition of Lyapunov transformation it has been assumed that the norm of the derivative of the matrix is bounded. This is actually not needed for preserving stability; it only guarantees that bounded matrices are mapped to bounded matrices in the transformation. Another definition used by Rugh, (1993) is: An $n \times n$ matrix $P(t)$ that is continuously differentiable and invertible at each t is called a Lyapunov transformation if there exist finite positive constants ρ and η such that for all t

$$\|P(t)\| \leq \rho, \quad |\det P(t)| \geq \eta \tag{7.17}$$

which is equivalent to finding a finite positive constant ρ such that

$$\|P(t)\| \leq \rho, \quad \left\|P^{-1}(t)\right\| \leq \rho \tag{7.18}$$

The fact that a Lyapunov transformation preserves the stability can easily be proved by noticing that for any non-singular $n \times n$ matrix A it holds

$$\left\|A^{-1}\right\| \leq \frac{\|A\|^{n-1}}{|\det A|} \tag{7.19}$$

where $\| \cdot \|$ denotes the spectral norm of a matrix, (Rugh, 1993)[1]. By using the stability results given in Section 3.2 in terms of the state-transition matrix, and approximating the norms of equations (7.8) and (7.9) by using the inequality gives the proof immediately.

The concept *reducibility* is defined to imply that a system matrix can be changed into a constant form by using a Lyapunov-transformation. More generally, two representations which are equivalent through a Lyapunov transformation are called *kinematically similar* by Harris and Miles (1980); the term *topologically equivalent realizations* is used by Tsakalis and Ioannou (1993). A well-known result in classical literature is that periodic systems are always reducible; see e.g. (Brockett, 1970), (Rugh, 1993).

[1] Actually, proving the inequality (7.19) has turned out to be a nice mathematical exercise to the enthusiasts!

7.3 z-invariant systems

To define a z-invariant system, one step further has to be taken: a system is z-invariant, if it has a representation, which, when represented as a function of some scale $z = f(t)$, is invariant with respect to the variable z.

It would be possible to modify all different definitions of 'time- invariant' systems such that 'z-invariant' systems would be defined analogously. However, for the current purposes it is enough to use two concepts viz. *zero state z-invariant* and (generally) *z-invariant systems*. Both of them are discussed in what follows.

A linear system is *z-invariant* (with respect to some fixed z), if its input-output behaviour can in z-domain be written as

$$\bar{y}(z) = F(z - z_0, 0)\bar{x}(z_0) + \int_{z_0}^{z} \bar{p}'(z - \xi, 0)\bar{u}(\xi)d\xi \tag{7.20}$$

for all initial times z_0, initial states $\bar{x}(z_0)$ and inputs $\bar{u}$. If the above representation is valid for certain initial states $\bar{x}(z_0)$ only, the system is z-invariant with respect to these states.

The equation simply states that a system is z-invariant, if it is zero state z-invariant and also 'invariant' with respect to the initial conditions. A z-invariant system is trivially zero state z-invariant.

The following results are easy to prove.

- If the system can be described by a z-invariant state-space realization, it is z-invariant.
- If the system is time-invariant, it is z-invariant also.
- If a z-invariant system has a state-space realization, this is not necessarily z-invariant.

To prove the first item note that a z-invariant state-space realization can be written as

$$\begin{aligned} \dot{x}(t) &= k(t)\bar{A}x(t) + k(t)\bar{B}u(t) \\ y(t) &= \bar{C}x(t) + \bar{D}u(t) \end{aligned} \tag{7.21}$$

with $x(t_0) = x_0$. Consider first the case $\bar{D} = 0$ meaning that the system is strictly proper. It follows that

$$p'(t, \tau) = \bar{C}\Phi(t, \tau)k(\tau)\bar{B} = k(\tau)\bar{C}e^{\bar{A}\int_{\tau}^{t} k(\nu)d\nu}\bar{B} \tag{7.22}$$

Using the transformation

$$z = f(t) = d_1 \int_{t_0}^{t} k(\nu) d\nu \tag{7.23}$$

gives

$$\frac{p'(t,\tau)}{\dot{f}(\tau)} = (1/d_1)\bar{C}e^{(1/d_1)\bar{A}d_1(\int_{t_0}^{t} k(\nu)d\nu - \int_{t_0}^{\tau} k(\nu)d\nu)}\bar{B} = (1/d_1)\bar{C}e^{(1/d_1)\bar{A}(z-\xi)}\bar{B} \tag{7.24}$$

which is a function of $z - \xi$ only so that the system is zero state z-invariant. Now, the zero input response of the system is

$$y_{zs}(t) = \bar{C}\Phi(t, t_0)x_0 = \bar{C}e^{\bar{A}\int_{t_0}^{t} k(\nu)d\nu}x_0 \tag{7.25}$$

In z-domain the result is

$$\bar{y}_{zs}(z) = \bar{C}e^{(1/d_1)\bar{A}z}x_0 \tag{7.26}$$

which is a function of $z - z_0 = z$ only. The system is z-invariant.

In the case that the matrix $\bar{D} \neq 0$ above, generalized functions have to be used in (7.22). Loosely speaking it means that the weighting function becomes

$$p'(t,\tau) = \bar{C}\Phi(t,\tau)k(\tau)\bar{B} + \bar{D}\delta_\tau(t) = k(\tau)\bar{C}e^{\bar{A}\int_{\tau}^{t} k(\nu)d\nu}\bar{B} + \bar{D}\delta_\tau(t) \tag{7.27}$$

where $\delta_\tau(t)$ is a unit impulse (Dirac delta function) entering at time τ. However, this kind of a representation is only formal, because the impulse is not a function in the sense of the classical function theory, and it is actually incorrect to speak about impulse functions at some particular time instant. The concept of *generalized functions* covered in *distribution theory* has to be used instead, see e.g. (Zadeh and Desoer, 1963), (Schwartz, 1951, 1957), (Beckenbach, 1961).

Let the transformation of the unit impulse $\delta_\tau(t)$ be $\bar{\delta}_\xi(z)$ in z-domain. Let $\varphi(t)$ be a test function, which can be differentiated an arbitrary number of times and which is identical to zero outside a finite interval (Zadeh and Desoer, 1963). In z-domain this test function is $\bar{\varphi}(z) = \bar{\varphi}(f(t)) = \varphi(t)$. In the following equation the transformation $z = f(t)$, $t = h(z)$ has been used, after which the selecting property of the impulse function is applied:

$$\int_{-\infty}^{\infty} \bar{\varphi}(z)\bar{\delta}_\xi(z)dz = \int_{-\infty}^{\infty} \varphi(t)\delta_\tau(t)\dot{f}(t)dt = \varphi(\tau)\dot{f}(\tau) \tag{7.28}$$

The result implies that $\bar{\delta}_\xi(z)$ is an impulse also (it has the selecting property), but it is not a unit impulse. The expression for it is

$$\bar{\delta}_\xi(z) = \dot{f}(\tau)\delta_\xi(z) = (dh(\xi)/d\xi)^{-1}\delta_\xi(z) \tag{7.29}$$

where $\delta_\xi(z)$ is the unit impulse in z-domain, which is assumed to enter at time $\xi = f(\tau)$. Now, if $\bar{\delta}_\tau(t)$ is the distribution in time-domain, which corresponds to $\delta_\xi(z)$, it follows:

$$\int_{-\infty}^{\infty} \varphi(t)\bar{\delta}_\tau(t)dt = \int_{-\infty}^{\infty} \bar{\varphi}(z)\delta_\xi(z)(dh(z)/dz)dz = \bar{\varphi}(\xi)(dh(\xi)/d\xi) \tag{7.30}$$

Again, $\bar{\delta}_\tau(t)$ is seen to be an impulse

$$\bar{\delta}_\tau(t) = (dh(\xi)/d\xi)\delta_\tau(t) = (\dot{f}(\tau))^{-1}\delta_\tau(t) \tag{7.31}$$

It is seen that the transformation of a unit impulse is an impulse with a different strength.

Solving for $\delta_\tau(t)$ from (7.31) and substituting into (7.27) gives

$$p'(t,\tau) = \bar{C}\Phi(t,\tau)k(\tau)\bar{B} + \bar{D}\delta_\tau(t) = k(\tau)\bar{C}e^{\bar{A}\int_\tau^t k(\nu)d\nu}\bar{B} + \bar{D}f'(\tau)\bar{\delta}_\tau(t) \tag{7.32}$$

But $\bar{\delta}_\tau(t) = \delta_\xi(z)$ for all τ, t, and the corresponding ξ, z so that

$$\frac{p'(t,\tau)}{\dot{f}(\tau)} = (1/d_1)\bar{C}e^{(1/d_1)\bar{A}(z-\xi)}\bar{B} + \bar{D}\delta_\xi(z) \tag{7.33}$$

and the system is zero-state z-invariant. (Note that $\delta_\xi(z)$, which is often written as $\delta(z-\xi)$ is a function of the difference $z-\xi$ only.)

For the proof of the second item, take $z = f(t) = at + b$, where the constants have been chosen such that $z > 0$ as $t \geq t_0$. Because the system is time-invariant, the weighting function can be represented as $p'(t,\tau) = p'(t-\tau, 0)$. It follows that

$$\frac{p'(t-\tau,0)}{\dot{f}(\tau)} = (1/a)p'(\frac{1}{a}(z-\xi))$$

where $\xi = f(\tau)$. For a time-invariant system the zero input response is a function of $t-t_0$ only. Because

$$t - t_0 = (1/a)(z - z_0)$$

the zero input response is also invariant with respect to z.

To show the validity of the last item an example can be used. Let a system with zero initial conditions be given by the weighting function

$$p'(t,\tau) = \tau^2 e^{t^3-\tau^3}$$

which cannot be written as a function of $t-\tau$ so that the system is time-varying. By using

$$z = f(t) = \int_{t_0}^t \nu^2 d\nu = (1/3)(t^3 - t_0^3)$$

it follows

$$\frac{p'(t,\tau)}{\dot{f}(\tau)} = e^{t^3-\tau^3} = e^{3(z-\xi)}$$

and the system is zero state z-invariant.

The weighting function can be written

$$p'(t,\tau) = \tau^2 e^{-\tau^3} e^{t^3}$$

so that a state-space representation for the system in question exists. Simple calculations show that the following two representations both describe the system

$$\dot{x}(t) = t^2 e^{-t^3} u(t)$$
$$y(t) = e^{t^3} x(t)$$

and

$$\dot{x}(t) = 3t^2 x(t) + t^2 u(t)$$
$$y(t) = x(t)$$

($x(t_0) = 0$ in both cases). The former representation is not z-invariant, while the latter one is.

A more practical example of a system having a z-invariant and a non-z-invariant representation is given next. Consider the process of two perfect mixers in series with varying flow rate $Q(t)$ but constant liquid volumes V_1 and V_2. As discussed in Chapter 3 the model of the system is given by equations (3.39) and (3.40) with k=0,1, $Q_0(t) = Q_1(t) = Q_2(t) = Q(t)$. The state representation is clearly z-invariant. However, by defining the state variables as $x_1(t) = c_2(t)$, $x_2(t) = \dot{c}_2(t)$, the state representation of the system becomes

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\frac{Q^2(t)}{V_2 V_1} & \frac{\dot{Q}(t)}{Q(t)} - \frac{Q(t)}{V_1} - \frac{Q(t)}{V_2} \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{Q^2(t)}{V_2 V_1} \end{bmatrix} u(t)$$
$$y(t) = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$$

where $u(t) = c_0(t)$, $y(t) = c_2(t)$. In order for the realization to be valid for the system, the differentiability of the flow rate function has to be assumed. The realization is not z-invariant, although it represents the same process.

The usefulness of the equivalence concepts presented becomes obvious with the following two propositions. Both of them assume linear systems that can be described by state-space equations. The first theorem is an extension to an assertion concerning time-invariant systems presented by Zadeh and Desoer (1963).

In the propositions and results it is assumed that a transformation function $f(\cdot)$ is fixed.

Proposition 6 *If a system is zero state z-invariant and zero state equivalent to a z-invariant system, it is then z-invariant with respect to all states that can be reached from the zero state.*

Proof: Let the weighting functions of the two systems be $\bar{p'_A}(z,\xi) = \bar{p'_A}(z-\xi,0)$, $\bar{p'_B}(z,\xi) = \bar{p'_B}(z-\xi,0)$, respectively. Because the systems are zero-state equivalent it holds

$$\int_{z_0}^{z} \bar{p'_A}(z-\xi,0)\bar{u}(\xi)d\xi = \int_{z_0}^{z} \bar{p'_B}(z-\xi,0)\bar{u}(\xi)d\xi \tag{7.34}$$

so that

$$\int_{z_0}^{z} (\bar{p'_A}(z-\xi,0) - \bar{p'_B}(z-\xi,0))\bar{u}(\xi)d\xi = 0 \tag{7.35}$$

Because this is valid for all $\bar{u}$ it follows

$$\bar{p'_A}(z-\xi,0) = \bar{p'_B}(z-\xi,0) \tag{7.36}$$

Consider the responses of the systems

$$\bar{y_A}(z) = \bar{F_A}(z,z_0)\bar{x}(z_0) + \int_{z_0}^{z} \bar{p'_A}(z-\xi,0)\bar{u}(\xi)d\xi \tag{7.37}$$

$$\bar{y_B}(z) = \bar{F_B}(z,z_0)\bar{x}(z_0) + \int_{z_0}^{z} \bar{p'_B}(z-\xi,0)\bar{u}(\xi)d\xi \tag{7.38}$$

where $\bar{x}(z_0)$ is reachable from the zero state. It is known that $\bar{F_B}(z,z_0) = \bar{F_B}(z-z_0,0)$. But it must also hold that $\bar{F_A}(z,z_0) = \bar{F_A}(z-z_0,0) = \bar{F_B}(z-z_0,0)$, because otherwise a control signal $\bar{u}$ could be chosen, which first drives the systems from zero state to $\bar{x}(z_0)$ and from there to some other states different for the systems A and B. But this is impossible, because the two systems were known to be zero-state equivalent by assumption. □

Proposition 7 *If a system is algebraically equivalent to a z-invariant system, it is z-invariant also.*

Proof: Consider two algebraically equivalent systems (7.2) and (7.4) so that (7.5) is valid. The weighting functions are identical so that the two systems are zero-state z-invariant. But the zero input responses

$$y_A(t) = C(t)\Phi_A(t,t_0)x_0 \tag{7.39}$$

$$\begin{aligned} y_E(t) &= G(t)\Phi_E(t,t_0)s_0 = C(t)P(t)P^{-1}(t)\Phi_A(t,t_0)P(0)P^{-1}(0)x_0 \\ &= C(t)\Phi_A(t,t_0)x_0 \end{aligned} \tag{7.40}$$

are also equivalent. If follows that if one of the systems is z-invariant, the other one is too. □

Two corollaries follow easily from the presented theorem.

Corollary 1 *If a system is algebraically equivalent to another system, which has a realization with constant coefficient matrices, the two systems are time-invariant and thus z-invariant also.*

Proof: The proof is an immediate consequence of the equivalence of the solutions of algebraically equivalent system realizations. □

Corollary 2 *If a system is algebraically equivalent to a system, which has a z-invariant realization, the system is z-invariant.*

Proof: A system with a z-invariant state-space representation is always z-invariant. The proof follows immediately by applying the Proposition 7. □

Example: Consider a system A, which has a state-space realization

$$\begin{aligned} \dot{x}(t) &= t^2 e^{-t^3} u(t) \\ y(t) &= e^{t^3} x(t) \end{aligned} \tag{7.41}$$

The weighting function of the system is $p'(t,\tau) = \tau^2 e^{t^3-\tau^3}$, and it was earlier shown that the system is zero-state z-invariant.

Representation

$$\begin{aligned} \dot{x}(t) &= 3t^2 x(t) + t^2 u(t) \\ y(t) &= x(t) \end{aligned} \tag{7.42}$$

can be considered to represent another system B, which has the same weighting function, and is therefore zero-state equivalent to system A. Now, the realization of B is z-invariant, so that the system is z-invariant. The system A is controllable, and so according to Proposition 6 it is then z-invariant with respect to all initial states. Hence, the system A is z-invariant irrespective of the initial state.

The same result follows also from Proposition 7 by noticing that the systems A and B are algebraically equivalent; the state transformation matrix is $P(t) = e^{-t^3}$.

Example: Consider a system with the representation

$$\begin{aligned} \dot{x}(t) &= -2tx(t) + e^{t-t^2} u(t) \\ y(t) &= 3e^{t^2-t} x(t) \end{aligned} \tag{7.43}$$

$(x(t_0) = x_0)$. The representation is not z-invariant. The weighting function of the system can easily be calculated to be $p'(t,\tau) = 3e^{-(t-\tau)}$, which shows that the system is zero state time-invariant. The use of a state transformation $P(t) = e^{t-t^2}$ gives

$$\begin{aligned} \dot{x}_1(t) &= -x_1(t) + u(t) \\ y(t) &= 3x_1(t) \end{aligned} \tag{7.44}$$

($x_1(t_0) = P^{-1}(t_0)x_0$). The new representation has constant coefficients so that it represents a time-invariant system. The system is time-invariant and z-invariant also.

The result may be difficult to believe by the form of the state equation. Indeed, the definitions of a time-invariant system vary in literature, and according to some definitions the above system would be called time-varying. Specifically, the behaviour of the state is dependent on that particular time instant, when the input is considered to enter. This alone would cause the system to be time-varying according to the other definitions mentioned. However, in the current text the definition previously given is used, which emphasizes the input-output behaviour of the system; this mapping behaves in an invariant manner irrespective of time.

7.4 State transformations and LQ optimal control

For comparison to the use of the modified time scale in analysis and synthesis, it is interesting to consider another approach, which is based on a direct state transformation in time domain. However, the idea is somehow analogous to the preceding discussion: how to change a linear but time-varying process model into a form more suitable for controller design by classical techniques. To this end, consider equations (7.2)-(7.5) in Section 7.1 again.

The idea is to use a state transformation with an invertible time varying matrix $P(\cdot)$. Through the transformation $x(t) = P(t)s(t)$ realizations (7.2) and (7.4) are equivalent. The weighting functions and input-output responses are equal. It is further known that if $P(\cdot)$ is a Lyapunov transformation, stability, controllability and observability remain invariant in the transformation. It follows that if realization (7.4) would be more suitable for controller design than (7.2), it would then be a good idea to use the changed realization.

An ideal situation would occur, if realization (7.4) would have constant coefficient matrices, such that control design could be carried out by using well-established methods of time-invariant systems. Unfortunately, this is not possible in general, because equations (7.5) cannot be solved for arbitrary desired matrices $E(\cdot)$, $F(\cdot)$, $G(\cdot)$ and $H(\cdot)$. But consider the following approach.

Consider the 'original' and 'target' systems

$$\begin{aligned} \dot{x}(t) &= A(t)x(t) + B(t)u(t), \quad x(t_0) = x_0 \\ y(t) &= C(t)x(t) \end{aligned} \tag{7.45}$$

$$\begin{aligned} \dot{s}(t) &= E(t)s(t) + F(t)v(t), \quad s(t_0) = s_0 \\ y(t) &= G(t)s(t) \end{aligned} \tag{7.46}$$

The real-valued time-varying coefficient matrices $A(\cdot)$, $E(\cdot)$ are $n \times n$ dimensional, $B(\cdot)$, $F(\cdot)$ are $n \times m$ dimensional and $C(\cdot)$, $G(\cdot)$ $r \times n$ dimensional. By applying *singular value decomposition* (SVD) to the functions $B(\cdot)$ and $F(\cdot)$

$$\begin{aligned} B(t) &= U_B(t)\Xi_B(t)V_B^T(t) \\ F(t) &= U_F(t)\Xi_F(t)V_F^T(t) \end{aligned} \tag{7.47}$$

it follows that

$$\begin{aligned} \dot{x}(t) &= A(t)x(t) + U_B(t)\bar{u}(t) \\ y(t) &= C(t)x(t) \end{aligned} \tag{7.48}$$

$$\begin{aligned} \dot{s}(t) &= E(t)s(t) + U_F(t)\bar{v}(t) \\ y(t) &= G(t)s(t) \end{aligned} \tag{7.49}$$

where

$$\begin{aligned} \bar{u}(t) &= \Xi_B(t)V_B^T(t)u(t) \\ \bar{v}(t) &= \Xi_F(t)V_F^T(t)v(t) \end{aligned} \tag{7.50}$$

In (7.47) the unitary (or even orthogonal, because they are real-valued) matrices $U_B(\cdot)$, $U_F(\cdot)$ and $V_B(\cdot)$, $V_F(\cdot)$ have the dimensions $n \times n$ and $m \times m$, respectively. The $n \times m$ dimensional matrices $\Xi_B(\cdot)$, $\Xi_F(\cdot)$ contain the singular values of $B(\cdot)$ and $F(\cdot)$ in descending order in their main diagonal; other elements are zero.

From equations (7.48) and (7.49) it is seen that the input variables $u(\cdot)$ and $v(\cdot)$ have been *augmented* to n-dimensional vectors, such that the coefficient matrices $U_B(\cdot)$ and $U_F(\cdot)$ are $n \times n$-dimensional and invertible. As mentioned, they are even orthogonal such that $U_B(t)^{-1} = U_B(t)^T$, $U_F(t)^{-1} = U_F(t)^T$. The idea in the augmentation of variables in the described way is in the fact that now the relationship between the original and target systems can be used in a mathematically useful way. To this end, consider the transformations

$$\begin{aligned} x(t) &= P(t)s(t) \\ \bar{u}(t) &= U(t)\bar{v}(t) \end{aligned} \tag{7.51}$$

where $P(\cdot)$ and $U(\cdot)$ are $n \times n$-dimensional matrices, which are invertible at each time instant. It then holds that

$$\begin{aligned} E(t) &= P^{-1}(t)\left[A(t)P(t) - \dot{P}(t)\right] \\ U_F(t) &= P^{-1}(t)U_B(t)U(t) \end{aligned} \tag{7.52}$$

If the coefficient matrices $E(\cdot)$ and $U_F(\cdot)$ in the target system are *chosen* and thus fixed, the transformation matrices are

$$\begin{aligned} P(t) &= \Phi_A(t,t_0)P(t_0)\Phi_E^{-1}(t,t_0) \\ U(t) &= U_B^{-1}(t)P(t)U_F(t) = U_B^T(t)P(t)U_F(t) \end{aligned} \tag{7.53}$$

which are invertible. The relationship between the augmented input and state variables of the original and target systems are then one-to-one through equation (7.51). Note that if the matrices $P(\cdot)$, $U_B(\cdot)$ and $U_F(\cdot)$ are Lyapunov transformations for each t, then $U(\cdot)$ is also a Lyapunov transformation. That implies that the state vector pairs $x(\cdot)$, $s(\cdot)$ and $\bar{u}(\cdot)$, $\bar{v}(\cdot)$ are compatible in the sense that there exist positive constants ρ_1, ρ_2, μ_1, and μ_2 such that $\|x(t)\| \leq \rho_1 \|s(t)\|$, $\|s(t)\| \leq \rho_2 \|x(t)\|$ and $\|\bar{u}(t)\| \leq \mu_1 \|\bar{v}(t)\|$, $\|\bar{v}(t)\| \leq \mu_2 \|\bar{u}(t)\|$.

For a control application, consider target system (7.49) and use the criterion

$$J_s(t_0) = \frac{1}{2}s^T(t_f)S(t_f)s(t_f) + \frac{1}{2}\int_{t_0}^{t_f} \left[s^T(\tau)X(\tau)s(\tau) + \bar{v}^T(\tau)R(\tau)\bar{v}(\tau)\right] d\tau \tag{7.54}$$

to be minimized. The optimal control is then

$$\bar{v}(t) = -L(t)s(t) \tag{7.55}$$

where

$$L(t) = R^{-1}(t)U_F^T(t)S(t) \tag{7.56}$$

and $S(\cdot)$ is obtained as a solution to the Riccati equation

$$-\dot{S}(t) = E^T(t)S(t) + S(t)E(t) - S(t)U_F(t)R^{-1}(t)U_F^T(t)S(t) + X(t) \tag{7.57}$$

with the boundary condition $S(t_f)$. In terms of the original system (7.48) the control signal is

$$\begin{aligned} \bar{u}(t) &= U(t)\bar{v}(t) = -U(t)L(t)s(t) \\ &= -U(t)L(t)P^{-1}(t)x(t) = -U_B^{-1}(t)P(t)U_F(t)L(t)P^{-1}(t)x(t) \\ &= -U_B^T(t)P(t)U_F(t)L(t)P^{-1}(t)x(t) \\ &= -U_B^T(t)P(t)U_F(t)R^{-1}(t)U_F^T(t)S(t)P^{-1}(t)x(t) \end{aligned} \tag{7.58}$$

and the closed loop state equation is given by

$$\begin{aligned} \dot{x}(t) &= \left[A(t) - U_B(t)U_B^T(t)P(t)U_F(t)R^{-1}(t)U_F^T(t)S(t)P^{-1}(t)\right] x(t) \\ &= \left[A(t) - P(t)U_F(t)R^{-1}(t)U_F^T(t)S(t)P^{-1}(t)\right] x(t) \end{aligned} \tag{7.59}$$

Criterion (7.54), which is minimized by using the optimal control, can be written in the form

$$\begin{aligned} J_x(t_0) &= \tfrac{1}{2}x^T(t_f)P^{-1}(t_f)^T S(t_f)P^{-1}(t_f)x(t_f) \\ &\quad + \tfrac{1}{2}\int_{t_0}^{t_f} \left[x^T(\tau)P^{-1}(\tau)^T X(\tau)P^{-1}(\tau)x(\tau) + \bar{u}^T(\tau)U^{-1}(\tau)^T R(\tau)U^{-1}(\tau)\bar{u}(\tau)\right] d\tau \end{aligned} \tag{7.60}$$

The optimal cost becomes

$$J_s^*(t_0) = \frac{1}{2} s^T(t_0) S(t_0) s(t_0) \tag{7.61}$$

which is the same as

$$\begin{aligned} J_x^*(t_0) &= \tfrac{1}{2} \left(P^{-1}(t_0)x(t_0)\right)^T S(t_0) \left(P^{-1}(t_0)x(t_0)\right) \\ &= \tfrac{1}{2} x^T(t_0) \left(P^T(t_0)\right)^{-1} S(t_0) P^{-1}(t_0) x(t_0) \end{aligned} \tag{7.62}$$

It is interesting to note that by choosing

$$\begin{aligned} &X(t) = P^T(t) X_2(t) P(t) \\ &R(t) = U^T(t) R_2(t) U(t) = U_F^T(t) P^T(t) U_B(t) R_2(t) U_B^T(t) P(t) U_F(t) \\ &S(t_f) = P^T(t_f) S_2(t_f) P(t_f) \end{aligned} \tag{7.63}$$

where $X_2(\cdot)$, $R_2(\cdot)$ and $S_2(t_f)$ are $n \times n$ dimensional weight matrices, criterion (7.60) changes into the familiar form

$$J_x(t_0) = \frac{1}{2} x^T(t_f) S_2(t_f) x(t_f) + \frac{1}{2} \int_{t_0}^{t_f} \left[x^T(\tau) X_2(\tau) x(\tau) + \bar{u}^T(\tau) R_2(\tau) \bar{u}(\tau) \right] d\tau \tag{7.64}$$

The question whether the resulting control law is asymptotically stable, is somewhat sophisticated. If the optimization horizon is finite, the value of the criterion is also finite, and the question of stability is actually not relevant. However, if the optimization horizon is infinite, the results on stability are usually restricted to the case of time-invariant systems in the literature (state-space representation with constant coefficient matrices, criterion with constant weight matrices). If the system and the criterion are time-varying, a reference is usually made to Kalman's work in early 60's, and even then the question of stability does not seem to be very clear. For references, see e.g. (Anderson and Moore, 1989), (Lewis and Syrmos, 1995).

If the system matrices A and B are constant, and the state and input weights in the criterion, X and R, are also constant, the question of stability is well-known. If the pair (A, B) is stabilizable and (A, C) detectable, where $C = X^T X$, the solution of the Riccati equation is unique and positive definite, and the resulting closed-loop system is asymptotically stable. (Stronger conditions than stabilizability and detectability are controllability and observability, which can also be used for sufficient conditions of asymptotic stability.)

Suppose that the 'original' time-varying state representation has been changed into a time-invariant 'target' representation with linear time-varying state and input variable changes, which are both Lyapunov transformations. If the criterion is set for the target system with constant weight matrices, the theory on stability can now be used. If the stabilizability and detectability conditions hold, the closed-loop target system is asymptotically stable. Therefore the original system is asymptotically stable also, because the variables are connected through Lyapunov transformations.

A drawback in the previous reasoning is that it is not at all certain that suitable Lyapunov transformations exist, that change a given system into a time-invariant form. Still, the strong connection between the two systems guarantees that if the other is stable, the other is too.

The above theories of optimal control seem nice, but a fundamental problem remains. The one-to-one relationship between an arbitrary target system and the given original system was obtained only by using augmented control variables. Control law (7.58) must be realized by a control signal of the original (real) system, i.e. considering equation (7.50). It must hold

$$\bar{u}(t) = \Xi_B(t) V_B^T(t) u(t) \tag{7.65}$$

But that equation does not generally have a solution for $u(\cdot)$ except in the rare case that $n = m$. Of course, an approximative solution could be tried

$$u(t) = \left(\Xi_B(t) V_B^T(t)\right)^\dagger \bar{u}(t) \tag{7.66}$$

where the notation $(\cdot)^\dagger$ means the pseudoinverse of a matrix. However, it has been shown by Montagnier *et al.* (2001) that this kind of an approximative solution does not necessarily give a good control response. In fact, the closed-loop system might even run unstable. Although the study of Montagnier *et al.* (2001) concerned linear time-periodic systems the difficulty might be fundamental for all time-varying systems as well.

Consider first the case $n = m$, so that $B(\cdot)$ is a square matrix, which is further assumed to be invertible at each time instant. Now the control signal can be realized exactly

$$u(t) = \left[\Sigma_B(t) V_B^T(t)\right]^{-1} \bar{u}(t) = V_B(t) \Sigma_B^{-1}(t) \bar{u}(t) \tag{7.67}$$

giving

$$\begin{aligned} u(t) &= -V_B(t)\Sigma_B^{-1}(t)U_B^T(t)P(t)U_F(t)R^{-1}(t)U_F^T(t)S(t)P^{-1}(t)x(t) \\ &= -B^{-1}(t)P(t)U_F(t)R^{-1}(t)U_F^T(t)S(t)P^{-1}(t)x(t) \end{aligned} \tag{7.68}$$

Note that in the criterion (7.60) it is now possible to choose $R(\cdot)$ as

$$R(t) = U_F^{-1}(t)P^T(t)B^{-1}(t)^T R_2(t)B^{-1}(t)P(t)U_F(t) \tag{7.69}$$

so that the criterion, which is minimized by the control law, becomes

$$J_x(t_0) = \frac{1}{2}x^T(t_f)S_2(t_f)x(t_f) + \frac{1}{2}\int_{t_0}^{t_f} \left[x^T(\tau)X_2(\tau)x(\tau) + u^T(\tau)R_2(\tau)u(\tau)\right] d\tau \tag{7.70}$$

The control law then obtains the final form

$$\begin{aligned} u(t) = &-B^{-1}(t)P(t)U_F(t)U_F^{-1}(t)P^{-1}(t)B(t)R_2^{-1}(t)B^T(t)P^{-1}(t)^T U_F(t)U_F^T(t) \\ &\cdot S(t)P^{-1}(t)x(t) = -R_2^{-1}(t)B^T(t)P^{-1}(t)^T S(t)P^{-1}(t)x(t) \end{aligned} \tag{7.71}$$

Note that in this particular case, where $n = m$, the above control law could have been calculated directly without any augmentations of the input variables. The starting point would be the system representations and transformations (7.2)-(7.5), and a LQ criterion in transformed variables, with the weight matrices chosen such that it becomes (7.70). The calculation is omitted here, because the case $n = m$ represents a rare and therefore relatively unimportant exception. However, note that the idea behind the use of the transformation of variables is to obtain a target system, in which the Riccati equation is easy to solve. For example, the target representation may be time-invariant.

Next, let $n > m$ and assume that in each time instant $rank(B(t)) = m$. Equation (7.65) does not generally have a solution, but an approximative solution that minimizes the Euclidean norm of the equation error is available. Because matrix $B(\cdot)$ was assumed to have "full" rank, the pseudoinverse has a simple matrix representation, see e.g. (Skogestad and Postlethwaite, 1996), and the approximative solution is

$$\begin{aligned} u(t) &= \left[\left(\Sigma_B(t)V_B^T(t)\right)^T\left(\Sigma_B(t)V_B^T(t)\right)\right]^{-1}\left(\Sigma_B(t)V_B^T(t)\right)^T\bar{u}(t) \\ &= \left[V_B(t)(\Sigma_B^T(t)\Sigma_B(t))V_B^T(t)\right]^{-1}\left(V_B(t)\Sigma_B^T(t)\right)\bar{u}(t) \\ &= V_B^{-1}(t)^T\left(\Sigma_B^T(t)\Sigma_B(t)\right)^{-1}\Sigma_B^T(t)\bar{u}(t) \\ &= V_B(t)\left(\Sigma_B^T(t)\Sigma_B(t)\right)^{-1}\Sigma_B^T(t)\bar{u}(t) \end{aligned} \tag{7.72}$$

But for the singular value decomposition it holds

$$\Sigma_B(t) = \begin{bmatrix} \Sigma_{1B}(t) \\ 0 \end{bmatrix};\ \Sigma_{1B}(t) = \mathrm{diag}\left(\sigma_{1B}(t)\cdots\sigma_{mB}(t)\right) \tag{7.73}$$

where

$$\mathrm{diag}\left(\sigma_{1B}(t)\cdots\sigma_{mB}(t)\right)$$

denotes the diagonal matrix with the singular values of $B(t)$ in the main diagonal. (Note that the singular values are positive, because $B(t)$ was assumed to have full rank.) Now it follows that

$$\begin{aligned} \Sigma_B^T(t)\Sigma_B(t) &= \begin{bmatrix} \Sigma_{1B}^T(t) & 0 \end{bmatrix}\begin{bmatrix} \Sigma_{1B}(t) \\ 0 \end{bmatrix} = \Sigma_{1B}^2(t) \\ &= \mathrm{diag}\left(\sigma_{1B}^2(t)\cdots\sigma_{mB}^2(t)\right) \end{aligned} \tag{7.74}$$

and the controller law obtains the form

$$u(t) = V_B(t)\left(\Sigma_{1B}^2(t)\right)^{-1}\Sigma_B^T(t)\bar{u}(t) \tag{7.75}$$

Finally, consider the case $m > n$ and assume that $rank(B(t)) = n$ at each time instant. Now (7.65) has an infinite number of solutions. The one with the minimal norm is given by the pseudoinverse

$$\begin{aligned} u(t) &= \left(\Sigma_B(t)V_B^T(t)\right)^T\left[\Sigma_B(t)V_B^T(t)\left(\Sigma_B(t)V_B^T(t)\right)^T\right]^{-1}\bar{u}(t) \\ &= V_B(t)\Sigma_B^T(t)\left[\Sigma_B(t)V_B^T(t)V_B(t)\Sigma_B^T(t)\right]^{-1}\bar{u}(t) \\ &= V_B(t)\Sigma_B^T(t)\left[\Sigma_B(t)\Sigma_B^T(t)\right]^{-1}\bar{u}(t) \end{aligned} \tag{7.76}$$

Again the singular values of $B(t)$ are known to be positive and it follows

$$\Sigma_B(t) = \begin{bmatrix} \Sigma_{1B}(t) & 0 \end{bmatrix}; \; \Sigma_{1B}(t) = \mathrm{diag}\left(\sigma_{1B}(t) \cdots \sigma_{nB}(t)\right) \tag{7.77}$$

and further

$$\begin{aligned} \Sigma_B(t)\Sigma_B^T(t) &= \begin{bmatrix} \Sigma_{1B}(t) & 0 \end{bmatrix} \begin{bmatrix} \Sigma_{1B}^T(t) \\ 0 \end{bmatrix} = \Sigma_{1B}^2(t) \\ &= \mathrm{diag}\left(\sigma_{1B}^2(t) \cdots \sigma_{nB}^2(t)\right) \end{aligned} \tag{7.78}$$

The final form of the control law is

$$u(t) = V_B(t)\Sigma_B^T(t)\left[\Sigma_{1B}^2(t)\right]^{-1}\bar{u}(t) \tag{7.79}$$

As an example, consider again the system of three perfect mixers in series, which was earlier discussed in Sections 6.1 and 6.3. The system is z-invariant with $A(t) = k(t)\bar{A}$, $B(t) = k(t)\bar{B}$, where $k(\cdot)$ denotes the flow rate. The target system (7.46) is chosen as $E(t) = \bar{A}$, $F(t) = \bar{B}$, which gives an interesting comparison to the technique of the modified time scale, because the 'target' system in both cases has constant coefficients. The state transformation matrix becomes

$$P(t) = \Phi_A(t, t_0)P_0\Phi_E^{-1}(t, t_0) \tag{7.80}$$

where

$$\Phi_A(t, t_0) = e^{\bar{A}\int\limits_{t_0}^{t} k(\nu)d\nu} \tag{7.81}$$

and

$$\Phi_E(t, t_0) = e^{\bar{A}(t-t_0)} \tag{7.82}$$

Choosing $P_0 = I$ and $t_0 = 0$ gives after some calculations

$$\begin{aligned} P(t) &= e^{\bar{A}\left[\int\limits_{0}^{t} k(\nu)d\nu - t\right]} \\ \det P(t) &= e^{\mathrm{tr}(\bar{A})\int\limits_{0}^{t}[k(\nu)-1]d\nu} \end{aligned} \tag{7.83}$$

where the well-known theorem of Abel-Jacobi-Liouville (see e.g. Harris and Miles, (1980)) has been used. The theorem states that for the state-transition matrix related to any system matrix $A(\cdot)$ it holds

$$\det\Phi_A(t, \tau) = e^{\int\limits_{\tau}^{t} \mathrm{trA}(\nu)d\nu}$$

Choosing again $k(t) = 1 + 0.5sin(0.1t)$ it is easy to see that the exponentials in $P(t)$ and $\det P(t)$ are bounded, so that $P(t)$ is a Lyapunov transformation.

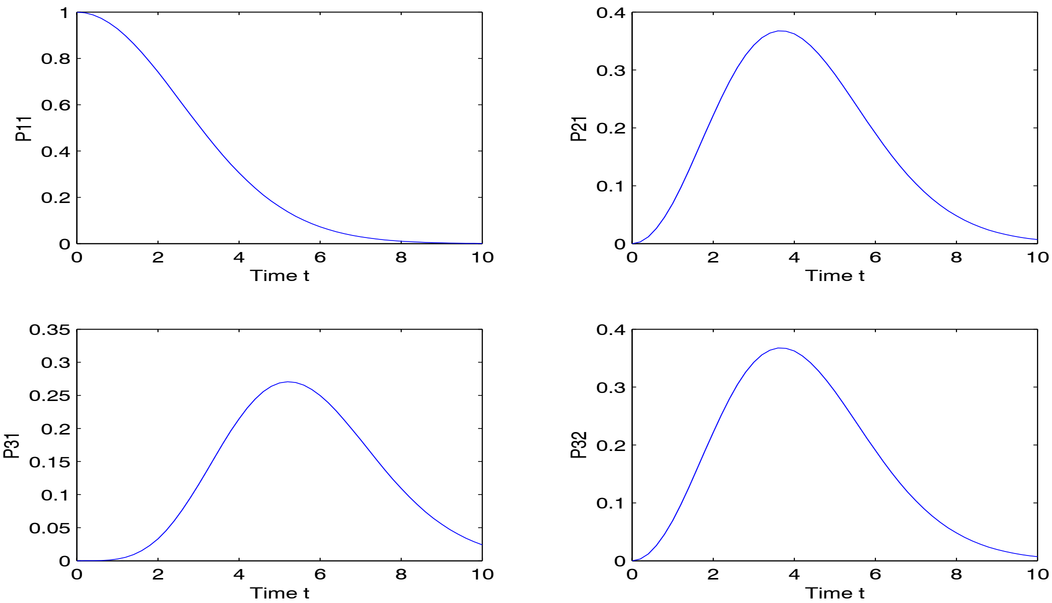

Figure 7.1: Some components of the transformation matrix $P(t)$

In Fig. 7.1 some components of matrix $P(\cdot)$ are shown. The matrix and its inverse stay bounded, although the values of the inverse matrix grow large as time approaches the value 10. The weight matrices in criterion (7.54) were chosen as

$$X = \begin{bmatrix} 0.01 & 0 & 0 \\ 0 & 0.01 & 0 \\ 0 & 0 & 0.01 \end{bmatrix}$$

$$R = \begin{bmatrix} 0.1 & 0 & 0 \\ 0 & 0.1 & 0 \\ 0 & 0 & 0.1 \end{bmatrix}$$

and control law (7.58) with the 'realization' (7.72) were used to control the process. A suitable static gain was added to the control loop to make the controlled system track the constant reference value 1. In Figs. 7.2 and 7.3 the simulation results have been presented. The dashed lines show the results obtained by the LQ_z controller described earlier in Section 6.3, while the solid line corresponds to the new time-variable control algorithm. The new controller establishes oscillations in the control signal at those time intervals, in which the function $P^{-1}(t)$ has extremely large values. This oscillation can be removed by limiting the values of $P^{-1}(t)$, which can be seen in Figs. 7.4 and 7.5. However, this cannot be regarded as a very beautiful solution. Although the performances of the z-controller and the new controller seem to be alike, the z-controller must be preferred, because the derivation of it did not contain any approximations or numerical tricks. On the other hand, the new time-variable controller covers a much larger set of processes,

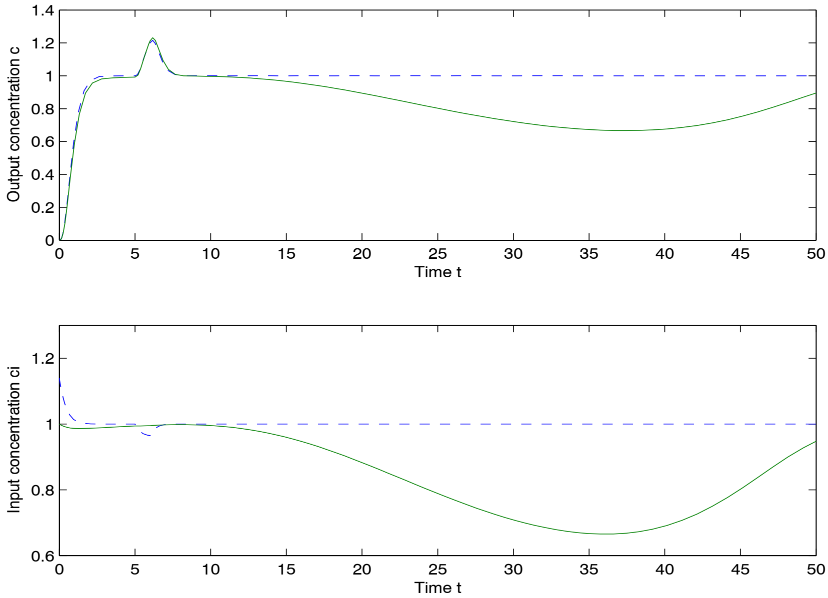

Figure 7.2: Closed loop performance for $t < 50$

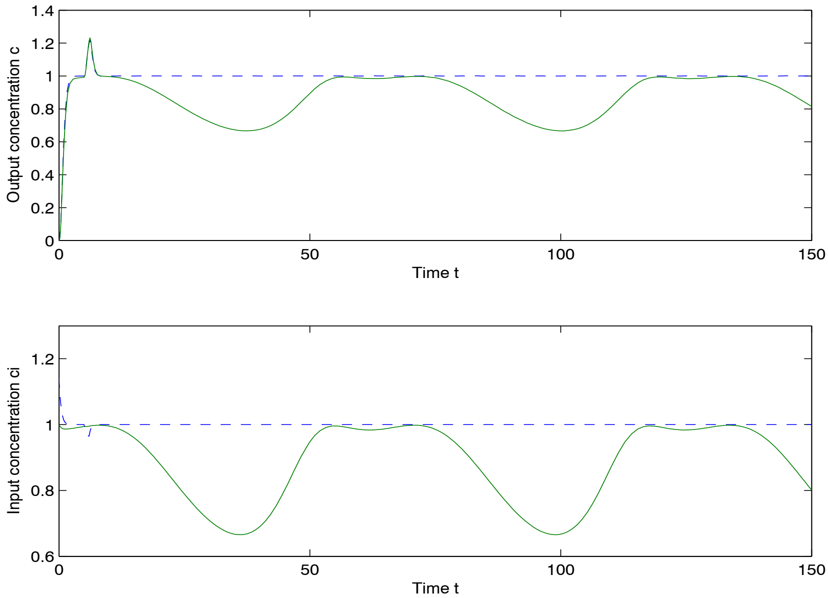

Figure 7.3: Closed loop performance for $t < 150$

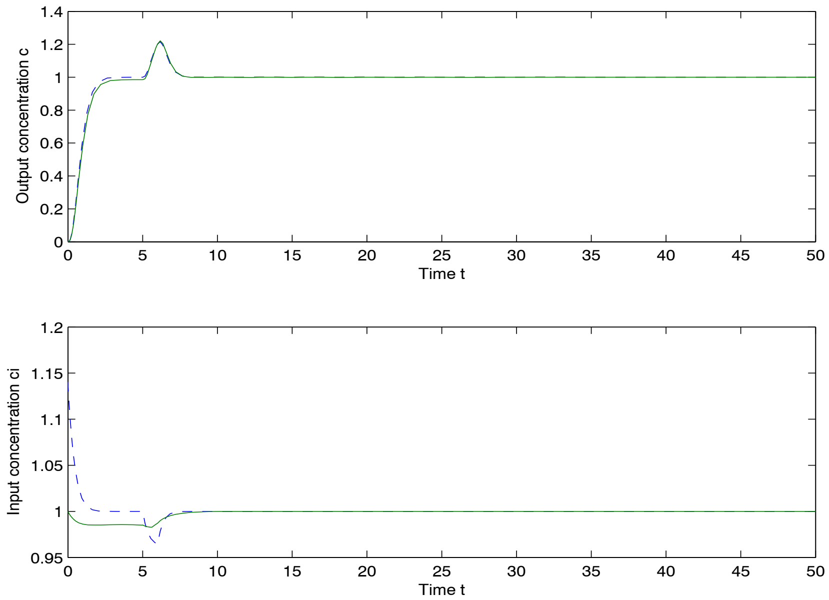

Figure 7.4: Closed loop performance for $t < 50$; modified control law

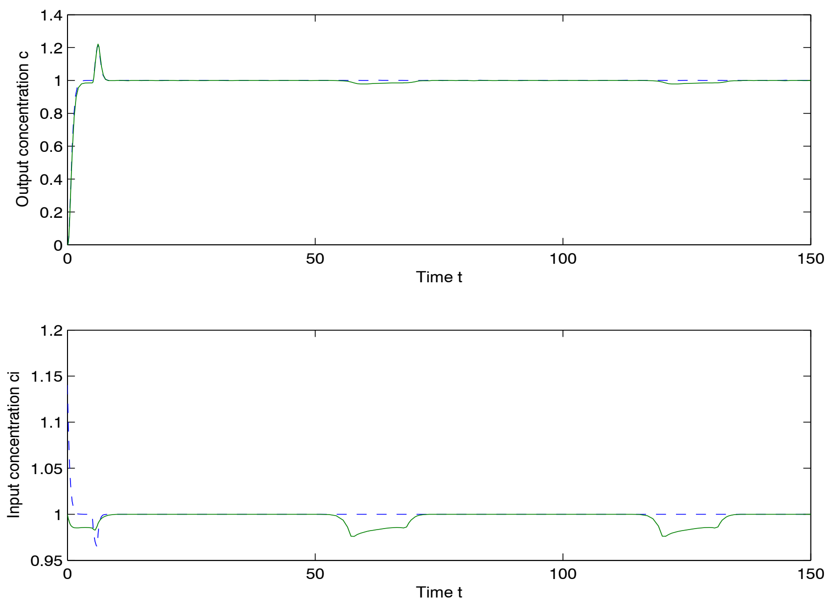

Figure 7.5: Closed loop performance for $t < 150$; modified control law

because they need not be z-invariant. It is evident that further research is needed in order to understand the new time-variable controller more deeply.

It should be noted that the method of using a Lyapunov transformation to change the realization into a more tractable form, and then doing controller design by using this target system, is an interesting but not the only available possibility. For example, in LQ design it is quite possible to use the time-varying equations as such to solve the control problem, as discussed in classical textbooks like Anderson and Moore (1989) or Lewis and Syrmos (1995), just to mention two examples. The drawback is that solving the time-varying equations is a hard problem, even numerically.

A more 'modern' approach would be to use robust control theory, H_2 and H_∞ methods, which also consider robustness issues of control design. The essentials of this approach are well documented in the literature, see e.g. Zhou and Doyle (1998). Robust control theory for linear time-varying systems is discussed e.g. in the interesting book by Ichikawa and Katayama (2001). However, the methods described in these references differ considerably from those presented in the current text, because they do not exploit any variable transformations, but merely consider the system equations as such and develop necessary analytical tools by extending the theory of time-invariant systems. The approach can be compared to that in Anderson and Moore (1989), but it is questionable, how practical the solutions are in reality. In the current text a more traditional and different method has been sought. There is no denying, however, that the results obtained are somewhat inconclusive.

Chapter 8

Conclusions

A systematic methodology for the analysis and controller design of a class of time variable systems has been developed in the text. It is well-known that nonlinear or time-variable systems are difficult to control, because their mathematical models are much more complex than those of linear time-invariant systems. For example, a general theory for the analysis and synthesis problems of linear but time variable systems is still lacking.

An important class of systems in the process industry deals with material transport, in which the liquid flow rates and volumes may be continuously varying. Often it is possible to describe these kinds of systems with linear models, in which the parameters are variable. The presented work has been an attempt to develop a general theory for these kinds of systems. The result is an extension to that known in the classical literature, where a modified time scale has been used to change the residence time distribution constant between different stationary operation constants. Now the theory has been extended to cover continuously varying operation conditions, viz. the case of varying flow rates in a material transport process. Also, the case with varying volumes has been discussed, and the related problems in this case have clearly been revealed by the developed mathematical methodology. Also, systematic controller design methods, which lead to time variable control algorithms, have been developed. The analysis has been strictly model-based in order to keep the development on a sound theoretical basis.

The background in analysis was in the modelling of material transport by using the residence time distribution theory and extending it to the time-varying case by letting the flow rates and liquid volumes be varying. The system theoretic connection of input-output behaviour was established by relating the RTD to the weighting function. The modified time scale (volumetric scale, z-scale) was then introduced to make the RTD and weighting functions equal. The concept of a z-invariant system was introduced to state conditions under which the model becomes invariant under the new scale making the analysis by classical methods possible. The 'scaled time variable' known in the classical literature was shown to be a special case in the new theory.

More structure was added in the development by using state-space representations and studying, whether a change of the time-variable leads to a similar simplification of the model as discussed earlier. It was demonstrated that this is indeed the case for models consisting of combinations of perfect mixers with possible recycling and bypass flows under variable flow rates. However, the question of variable liquid volumes turned out to be a much more difficult problem, which was to be expected based on the previous analysis also. There are computational ways to deal with the case of variable volume, but a real solution, which would be mathematically as beautiful as the previous case, was never found.

The discussion was then extended to cover time delays, which are naturally modelled by using plug flow vessels. The concept of the delay function was introduced, and it was shown that time-varying delay changes into a constant in the volumetric scale; however, again the assumption of a variable flow rate but constant liquid volume was needed.

The developed theory was tested by using a laboratory scale pilot plant with different process vessels. Both chemical and radioactive tracers were used under changing flow rates and volumes in the determination of the time-variable residence time distribution. The unification of the RTDs predicted by the theory was verified by excellent results in the tests. Some problems were met by one particular vessel, which had a very different velocity profile under different flow rates. That was the result of bad design in the construction of the vessel, and it was unfortunate that the case of varying liquid volume could only be tested by another vessel, which behaved almost like a perfect mixer.

Systematic controller design methods were then developed to demonstrate the practical applicability of the theory. PID controllers were discussed, and it was shown how a controller with time varying parameters resulted by assuming a z-invariant system model and using a normal PID controller in z-domain. The stability of the closed loop system was shown to be a direct consequence of the design method. The performance and the particular stability result was demonstrated by both simulation and by using the pilot plant. An LQ controller was also developed and its operation was tested by simulation.

The developed theory is restricted to z-invariant systems, and although this class is important in process industry, the boundaries encountered for example in the case of varying liquid volumes are not pleasant. Therefore an effort was made to establish a totally different but more general way to control linear time-varying systems. This new concept is based on a direct transformation of the state and input variables to change the model amenable to analysis and control, whereafter the control of the original process becomes straightforward. Even if techniques related to 'the change of variables' is not unfamiliar in control theory generally, the discussion in this text concerning variable changes in input-output models is believed to be new. The results turned out to be promising and interesting, but there is no denying the fact that there are difficult problems to bring this approach into a general and mature design technique. Fundamentally, this is because time-variable systems are analytically much more difficult to deal with than time-invariant systems or some particular classes of time-variable systems.

Bibliography

Anderson, B. D. O. and J. B. Moore (1989). *Optimal Control*, (Linear Quadratic Methods), Prentice-Hall, Inc.

Andersson, T. and P. Pucar (1995). Estimation of Residence Time in Continuous Flow Systems with Dynamics. *Journal of Process Control*, Vol. 5, No.1, pp. 9–17.

Balchen, J. G., Lie, B. and I. Solberg (1988). Internal Decoupling in Non-Linear Process Control. *Modeling, Identification and Control*, Vol. 9, pp. 137–148.

Bartle, R. G. and C. I. Tulcea (1970). *Honors Calculus.* Scott, Foresman and Company, Glenview, Illinois.

Bazin, C. and D. Hodouin (1988). Residence time distributions modelling for mineral processes evaluation. *CIM Bulletin*, Vol. 81, No. 911, pp. 115–125.

Beckenbach, E. F. (ed.) (1961). *Modern Mathematics for the Engineer*, Second Series, chap1, "From Delta Functions to Distributions", by A. Erdelyi, McGraw-Hill Book Company, Inc., New York.

Bialkowski, W. L. (1993). Dreams versus reality: A view from both sides of the gap. *Pulp and Paper Canada*, Vol 94, No 11.

Blomberg, H. and R. Ylinen (1983). *Algebraic Theory for Multivariable Linear Systems.* Academic Press.

Brockett, R. W. (1970). *Finite Dimensional Linear Systems.* John Wiley & Sons, Inc.

Chen, C-T. (1999). *Linear System Theory and Design.* Oxford University Press.

Coddington, E. A. and N. Levinson (1955). *Theory of Ordinary Differential Equations*, McGraw-Hill, New York.

Danckwerts, P. V. (1953). Continuous Flow Systems: Distribution of Residence Times. *Chemical Engineering Science*, No 2, pp. 1–18.

Denn, M. (1986). *Process Modeling.* John Wiley & Sons, Inc.

Dickens, A. W., M. R. Mackley and H. R. Williams (1989). Experimental residence time distribution measurements for unsteady flow in baffled tubes. *Chemical Engineering Science*, 44, pp. 1471–1479.

Fernández-Sempere, J., R. Font-Montesinos and O. Espejo-Alcaraz (1995). Residence Time Distribution for Unsteady-State Systems. *Chemical Engineering Science*, Vol. 50, No 2, pp. 223–230.

Górecki, H., S. Fuksa, P. Grabowski and A. Korytowski (1989). *Analysis and Synthesis of Time Delay Systems.* John Wiley & Sons, PWN-Polish Scientific Publishers, Warszawa.

Guizerix, J. (Ed). (1990). *Guidebook on Radioisotope Tracers in Industry.* International Atomic Energy Agency, Vienna.

Harris, C. J. and J. F. Miles (1980). *Stability of Linear Systems.* Academic Press, London.

Henson, M. A. and D. E. Seborg (1990). Input-Output Linearization of General Nonlinear Processes. *AIChE Journal*, Vol. 36, pp. 1753–1757.

Himmelblau, D. M. and K. B. Bischoff (1968). *Process Analysis and Simulation: Deterministic Systems.* John Wiley & Sons, Inc.

Häggblom, K. E. (1993). Experimental Comparison of Conventional and Nonlinear Model-Based Control of a Mixing Tank. *Industrial & Engineering Chemistry Research*, Vol. 32, pp. 2653–2661.

Ichikawa, A. and H. Katayama (2001). *Linear Time-Varying Systems and Sampled-data Systems*. Springer-Verlag London Limited.

Isaksson, A. (1993). Estimation of Average Residence Time Given an Identified Arx-Model. *Proceedings of the European Control Conference*, Groningen, The Netherlands, pp. 1336–1341.

Jamshidi, M. and M. Malek-Zavarei (1986). *Linear Control Systems, (A Computer Aided Approach).* Pergamon Books Ltd.

Jutila, P. (1983). An Application of Adaptive pH-Control Algorithms based on Physico-Chemical Modelling in a Chemical Waste-Water Treatment Plant. *International Journal* *of Control*, Vol. 38, No. 3, pp. 639–655.

Jutila, P. and P. Jaakola (1986). *Tests with Five Adaptive pH-Control Methods in Laboratory.* Helsinki University of Technology, Control Engineering Laboratory, Report 65.

Jutila, P., H. Hyötyniemi and J-P Ylén (1999). Problems and practice of pH process modelling and control in stirred tank reactors. *Archives of Control Sciences*, Vol. 9(XLV), No. 3-4, pp. 5–31.

Kravaris, C. and M. Soroush (1990). Synthesis of Multivariable Nonlinear Controllers by Input/Output Linearization. *AIChE Journal*, Vol. 36, pp. 249–264.

Levenspiel, O. (1962). *Chemical Reaction Engineering.* John Wiley & Sons, Inc.

Lewis, F.L. and V.L. Syrmos (1995). *Optimal Control.* Second edition. John Wiley & Sons, Inc.

Lyapunov, A. M. (1966). *Stability of Motion.* Academic Press, New York.

Marino, R. and P. Tomei (1995). *Nonlinear Control Design.* Prentice Hall.

Markus, L. (1955). Continuous Matrices and the Stability of Differential Systems. *Math. Zeitschr.*, Vol. 62, pp. 310–319.

Mohler, R. (1991). *Nonlinear Systems*, Vol 1. Dynamics and Control. Prentice-Hall, Inc.

Montagnier, P., R. J. Spiteri and J. Angeles (2001). Pitfalls of a least-squares-equivalent controller for linear, time-periodic systems. *International Journal of Control*, Vol. 74, No. 2, pp. 199–204.

Najim, K., A. S. Poznyak and E. Ikonen (1996). Calculation of residence time for nonlinear systems. *International Journal of Systems Science*, Vol. 27, no. 7, pp. 661–667.

Narendra, K.S. and A.M. Annaswamy (1989). *Stable Adaptive Systems.* Prentice-Hall, Inc.

Nauman, E.B. (1969). Residence Time Distribution Theory for Unsteady Stirred Tank Reactors. *Chemical Engineering Science*, Vol. 24, pp. 1461–1470.

Nemytskii, V. V. and V. V. Stepanov (1960.) *Qualitative Theory of Differential Equations.*, Princeton University Press, Princeton, NJ.

Niemi, A. (1977a). Residence Time Distributions of Variable Flow Processes. *Interna-*

tional Journal of Applied Radiation and Isotopes, Vol 28, pp. 855–860.

Niemi, A. and P. Jutila (1977b). pH control by linear algorithms. *Preprints of the 5th IFAC/IFIP Conf.*, Hague, The Netherlands, pp. 289–297.

Niemi, A. (1981). Invariant Control of Variable Flow Processes. *Proceedings of the 8th IFAC World Congress*, Kyoto, Japan, pp. 2687–2692.

Niemi, A. (1988). Variable Parameter Model of the Continuous Flow Vessel. *Mathl Comput. Modelling*, Vol 11, pp. 32–37.

Niemi, A. J. (1990). Tracer Responses and Control of Vessels with Variable Flow and Volume, *Isotopenpraxis*, Vol. 26, No. 9, pp. 435–438.

Niemi, A. J., K. J. Åström., E. Boje., H. A. Spang and E. Gottzein (1990). Application of Gain Scheduling. *Proceedings of the 11th IFAC World Congress*, Tallinn. (Discussion Session).

Niemi, A. J. (1991). Process Control under Variable Flow and Volume. *Proceedings of the IFAC Int. Symp. on Intelligent Tuning and Adaptive Control (ITAC 91)*, Singapore.

Niemi, A. J., L. Tian, P. Jutila and R. Ylinen (1993). Experimental Identification of Variable Parameter Flow Processes. *Proceedings of the 12th IFAC World Congress*, Sydney, Australia, pp. 697–702.

Niemi, A. J., L. Tian, R. Ylinen and K. Zenger (1997). Robust Control of Variable Flow Processes. *Proceedings of the 2nd IFAC Workshop on New Trends in Design of Control Systems*, Smolenice, Slovak Republic, pp. 93–98.

Niemi, A. J., K. Zenger, J. Thereska and J. G. Martinez (1998). Tracer Testing of Processes Under Variable Flow and Volume. *Nukleonika*, Vol. 43, No. 1, pp. 73–94.

Niemi, A. J., K. Zenger and P. Viitanen (2001). Incorporation of Tracer Test Information Into the Control Loop Design of a Continuous Flow Process. *Proceedings of the First International Congress Tracer and tracing methods*, Nancy, France, pp. 221–228.

Nihtilä, M. (1991). Finite Pole Assignment for Systems with Time-Varying Input Delays. *Proceedings of the 30th IEEE Conference on Decision and Control*, Brighton, United Kingdom, pp. 927–928.

Nir, A. (1973). Tracer Relations in Mixed Lakes in Non-Steady State. *Journal of Hydrology*, Vol. 19, pp. 33–41.

Orava, P. J. (1973). Causality and State Concepts in Dynamical Systems Theory. *Int. J. Systems Sci.*, Vol. 4, No. 4, pp. 679–691.

Orava, P. J. (1974). Notion of Dynamical Input-Output Systems: Causality and State Concepts. *Int. J. Systems Sci.*, Vol. 5, No. 8, pp. 793–806.

Padulo, L. M. and A. Arbib (1974). *System Theory.* W. B. Saunders Company.

Rugh, W., J.(1993). *Linear System Theory.* Prentice-Hall, Inc.

Schwartz, L.(1951, 1957). *Théorie des distributions*, vols. 1 and 2, Hermann & Cie., Paris.

Seinfeld, J. H. and L. Lapidus (1974). *Mathematical Methods in Chemical Engineering.* Vol 3, Prentice-Hall, Inc.

Thereska, J., J. P. Leclerc, E. Plasari and J. Villermaux (1996). Tracer experiments and residence time distributions in the analysis of industrial units: case studies. *Proceedings of the 12th International Congress of Chemical and Process Engineering*, (CHISA'96), Praha, Czech Republic.

Thereska, J. (2001). Research and development in radiotracer methodology and technology. *Proceedings of the First International Congress Tracer and tracing methods*, Nancy, France, pp. 335–345.

Tian, L., A. J. Niemi and R. Ylinen (1992). Recursive Process Identification Under Variable Flow and Volume, *Selected papers from the 3rd IFAC Symposium of Dynamics and Control of Chemical Reactors, Distillation Columns and Batch Processes (DYCORD+'92)*, Maryland, USA, pp. 339–344.

Tian, L. (1994). *The Identification and Robust Control of Time Variable Flow Processes: A FIR Approach.* Acta Polytechnica Scandinavica, Mathematics and Computing in Engineering Series No. 67. (Doctoral Thesis.)

Tsakalis, K. S. and P. A. Ioannou (1993). *Linear Time-Varying Systems.* Prentice-Hall, Inc.

Wiberg, D. M. (1971). *State Space and Linear Systems.* McGraw-Hill, Inc.

Wigren, T. (1990). *Recursive identification based on the nonlinear Wiener model.* ACTA Universitatis Upsaliensis, no 31, Department of Technology, Uppsala University.

Willems, J. (1970). *Stability Theory of Dynamical Systems.* Thomas Nelson.

Ylinen, R. (1980). *An Algebraic Theory for Analysis and Synthesis of Time-Varying Linear Differential Systems.* (Doctoral thesis), Acta Polytechnica Scandinavica, Mathematics and Computer Science Series No. 32.

Zadeh, L. A. and C. A. Desoer (1963). *Linear System Theory–The State Space Approach.* McGraw-Hill, New York.

Zenger, K. (1992). *Analysis and Control Design of a Class of Time-Varying Systems.* Helsinki University of Technology, Control Engineering Laboratory, Report 88.

Zenger, K. (1993). Analysis and Control of Time-Varying Flow Processes by Using the State-Space Approach. *Proceedings of the 12th World Congress of IFAC*, Sydney, Australia, pp. 43–48.

Zenger, K. and R. Ylinen (1994). Simulation of Variable Delays in Material Transport Models. *Mathematics and Computers in Simulation*, Vol. 37, pp. 57–72.

Zenger, K. (1995). Time-Variable Models for Mixing Processes Under Unsteady Flow and Volume. *Proceedings of the 4th IFAC Symposium of Dynamics and Control of Chemical Reactors, Distillation Columns and Batch Processes (DYCORD+'95)*, Helsingør, Denmark, pp. 57-62.

Zenger, K., A. J. Niemi and M. Martikainen (1996). Analysis and Pilot Scale Testing of a Class of Process Control Algorithms with Variable Parameters. *Proceedings of the 13th IFAC World Congress*, San Francisco, USA. (CD-ROM format, in preprints Vol. M.)

Zhou, K. and J. C. Doyle (1998). *Essentials of Robust Control.* Prentice-Hall Inc.

Žitný, R. and J. Thýn (1996). *Residence Time Distribution Software Analysis.* User's manual. International Atomic Energy Agency.

Åström, K. J. and T. Hägglund (1995). *PID Controllers: Theory, Design, and Tuning.* (2nd Edition.) Instrument Society of America.

Åström, K. J. and B. Wittenmark (1995). *Adaptive Control.* (2nd edition.) Addison-Wesley Publishing Company.

Åström, K. J. and B. Wittenmark (1997). *Computer-Controlled Systems, Theory and Design*, Third edition, Prentice-Hall Inc.

HELSINKI UNIVERSITY OF TECHNOLOGY CONTROL ENGINEERING LABORATORY

Editor: H. Koivo

Report 124 Uykan, Z.
Clustering-Based Algorithms For Radial Basis Function and Sigmoid Perceptron Networks. June 2001.

Report 125 Hyötyniemi, H.
Multivariate Regression - Techniques and tools. July 2001.

Report 126 Kaartinen, J.
Data Acquisition and Analysis System for Mineral Flotation. October 2001.

Report 127 Ylén, J.-P.
Measuring, Modelling and Controlling the pH value and the Dynamic Chemical State. November 2001.

Report 128 Gadoura, I. A., Suntio, T.
Implementation of Optimal Output Characteristic for a Telecom Power Supply - Fuzzy-logic approach. April 2002.

Report 129 Elmusrati, M. S.
Power Control and MIMO Beamforming in CDMA Mobile Communication Systems. August 2002.

Report 130 Pöyhönen, S., Negrea, M., Arkkio, A., Hyötyniemi, H.
Comparison of Reconstruction Schemes of Multiple SVM's Applied to Fault Classification of a Cage Induction Motor. August 2002.

Report 131 Pöyhönen, S.
Support Vector Machines in Fault Diagnostics of Electrical Motors. September 2002.

Report 132 Gadoura, I. A.
Design of Robust Controllers for Telecom Power Supplies. September 2002.

Report 133 Hyötyniemi, H.
On the Universality and Undecidability in Dynamic Systems. December 2002.

Report 134 Elmusrati, M. S., Koivo, H. N.
Radio Resource Scheduling in Wireless Communication Systems. January 2003.

Report 135 Blomqvist, E.
Security in Sensor Networks. February 2003.

Report 136 Zenger, K.
Modelling, Analysis and Controller Design of Time-Variable Flow Processes. March 2003.

ISBN 951-22-6438-2

ISSN 0356-0872

Picaset Oy, Helsinki 2003